On Families of Elliptic Curves

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“Happy families are all alike.”

–Leo Tolstoy, *Anna Karenina*

Declaration

The work in this thesis is my own, except where otherwise stated.

Brendan Fong

Acknowledgements

It is not inappropriate that I should write this as the dawn begins to light my window. This thesis is the culmination of the last 40 months of mathematical education, and over this period the entirety of many a night has been spent, as this last one has been, thinking about mathematics. As much as it is dedicated to anybody, this thesis is for those who have joined me during these extended evenings. They've been fun.

Over the last few years I have had the privilege of learning from many great teachers and students of mathematics. All these people I thank. Of them, however, I owe special thanks to my thesis supervisor Dr James Borger. I thank him for suggesting a fascinating thesis topic, and his support through the hard work. His generosity with his time and energy, in particular over the hectic past few weeks, has been far beyond his duty. I also acknowledge the support of Professor Amnon Neeman, Professor Michael Eastwood, Associate Professor Alex Isaev and Dr Elizabeth Ormerod, both mathematical and otherwise, over this past year.

Of the students, I thank in particular my company in my (adopted) office: David, Fiona, Howard and Tom. The writing of this thesis has not been easy, but they have made the long hours at a computer more bearable, and helped out as much as they could. I also am grateful to Daniel, Jono and Alex for providing detailed feedback on parts of a draft, and Jiapei, Patrick, Bonnie, Hye Sun and Simba for remembering to check up on my sanity even now and then.

Finally, I thank my brothers Justin and Calvin, and my parents. Despite the distance, their continued support has meant so much to me.

Abstract

This essay introduces the moduli problem for elliptic curves over the complex numbers. Taking a complex geometric perspective, we show that this moduli problem is equivalent to the moduli problem for lattices and that no moduli space, as a complex manifold, exists. In lieu of this we discuss the construction of a coarse moduli space, and the use of level structures in rigidifying the moduli problem.

Contents

Preface

The study of elliptic curves dates back at least to Euler's work on elliptic integrals and yet, due to both intrinsic beauty and significant applications, still forms an active area of research today. One method of studying elliptic curves is to examine their behaviour under infinitesmial deformation. Following the work of Kodaira and Spencer on deformations of complex manifolds, these deformations may be captured by so-called families of elliptic curves. The focus of this essay is the classification of such families.

We begin, in Chapter 0, by introducing some basic notions in the theory of complex manifolds and Riemann surfaces. This allows us to give a precise definition of elliptic curves and of their families, and equips us with the basic tools we shall use to understand them.

Chapter 1 then discusses moduli problems. Through this language we specify the sense in which we wish to classify families of elliptic curves: we wish to construct a family of elliptic curves from which every other family arises in a natural way. We call such a family a 'universal family'. In this chapter we also consider the example moduli problem of lattices, noting that it is not solvable due to the existence of nontrivial automorphisms.

In Chapter 2 we begin work on the moduli problem for elliptic curves by reducing it the problem to that of lattices. As we noted in Chapter 1, however, the moduli problem for lattices is not solvable. We are thus led to consider methods of weakening the moduli problem.

The first of these, seen in Chapter 3, is the construction of a coarse moduli space, which classifies all elliptic curves and their local deformations, but does not have a universal family. The second, seen in Chapter 4, is that of level structures. This rigidifies elliptic curves slightly, removing their nontrivial automorphisms, and hence allowing the construction of a universal family for these elliptic curves with extra structure.

The background assumed in this thesis was in the end chosen to be no more than the material taught in the undergraduate curriculum at the ANU. More precisely, this thesis is targeted at the reader who has taken first courses in complex analysis, algebraic topology and differential geometry, but may not be familiar with constructions such as Riemann surfaces and fibre bundles.

The mathematics developed here is well-known, and I can claim no originality to the broad ideas contained within it. General references have been given as appropriate in each section, and where arguments have been taken from sources they have been specifically cited as such. Furthermore, the general perspective of this thesis owes much to discussions with my supervisor Dr James Borger, as do many of the details in Chapter 4. What little that remains, however, I claim as my own work.

Chapter 0

Preliminaries

The aim of this essay is to understand elliptic curves and their families. In this chapter we make a start by defining what these are.

0.1 Background on Complex Manifolds

Elliptic curves historically arise as the surfaces that form the natural domain of integration when computing the arc-length of an ellipse. This perspective does not interest us so much in our discussion of families of elliptic curves, although we will see glimpses of it in later chapters. For now, we introduce them as tori with a complex structure. A more detailed introductory account of complex manifolds can be found in Fritzsche and Grauert [10, Ch.IV] and Kodaira [20, Ch.2], while Farkas and Kra [8] and Forster [9] provide excellent expositions of the material specific to Riemann surfaces. Many results in this section are stated with only a reference given for proof, the exceptions being those results for which ideas in their proof will be relevant later.

Complex Manifolds

An n-dimensional complex manifold is a space that is locally n-dimensional complex space. Formally, this means the following. Let X be a Hausdorff topological space. An *n-dimensional (complex) coordinate neighbourhood* (U, z) in X consists of an open set U of X and a homeomorphism z of U onto an open subset of $\mathbb{C}^n$. Two n-dimensional coordinate neighbourhoods (U_i, z_i), (U_j, z_j) are *(holomorphically) compatible* if the function

$$w_{ij} := z_i \circ z_j^{-1} : z_j(U_i \cap U_j) \longrightarrow z_i(U_i \cap U_j)$$

is biholomorphic. A countable covering of X by a collection of pairwise compatible n-dimensional coordinate neighbourhoods of X is called an *n-dimensional complex atlas* on X. Two complex atlases are equivalent if their union is also a complex atlas. An *n-dimensional complex structure* on X is an equivalence class of atlases.

Definitions 0.1. An *n-dimensional complex manifold* is a Hausdorff topological space X with an n-dimensional complex structure. A *Riemann surface* is a connected 1-dimensional complex manifold.

We make the observation that a 0-dimensional complex manifold is simply a discrete set.

Let $\mathcal{U} = (U_i, z_i)_{i\in I}$ be an n-dimensional coordinate covering of a Hausdorff space X. A function $f : U \to \gamma$ defined on an open subset U of X is called *holomorphic relative to $\mathcal{U}$* if for all $i \in I$

$$f \circ z_i^{-1} : z_i(U \cap U_i) \longrightarrow \mathbb{C}^n$$

is holomorphic. If X is a complex manifold and this is true with respect to all coordinate neighbourhoods defining the complex structure on X, the f is called *holomorphic*. More generally, we define holomorphic functions between any two complex manifolds as follows.

Definition 0.2. Let X, Y be complex manifolds. A map $f : X \to Y$ is called *holomorphic* if for every $p \in X$ there exists a coordinate neighbourhood (U, z) in X at p and a coordinate neighbourhood (V, w) in Y at $f(p)$ with $f(U) \subset V$ such that

$$w \circ f \circ z^{-1} : z(U) \longrightarrow w(V)$$

is holomorphic. If the inverse function f^{-1} exists and is also holomorphic we call f *biholomorphic.*

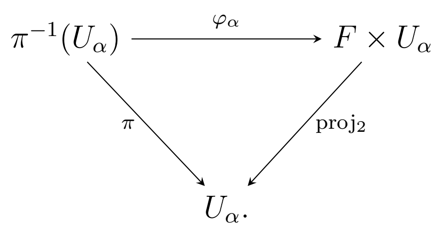
We may also view manifolds as ringed spaces. Given an open set U of $\mathbb{C}^n$, the sheaf $\mathcal{O}_U$ of holomorphic functions on U makes $(U, \mathcal{O}_U)$ a ringed space. We shall call such a ringed space a *standard ringed space.* Let $\{(U_i, z_i)\}_{i\in I}$ be a coordinate covering of a complex manifold X. The holomorphic functions on a complex manifold X then form a sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras on X, and this makes X into a ringed space locally isomorphic to a standard ring space. In fact any such sheaf of $\mathbb{C}$-algebras—that is, any sheaf such that there exists an open cover $\{U_i\}_{i\in I}$ of X with each $(U_i, \mathcal{O}_X|_{U_i})$ a standard ring space—defines a complex structure on X, and uniquely so.

Complex manifolds, together with holomorphic maps, form a category; we denote this category $\mathbf{Mfld}_{\mathbb{C}}$.

Fibre Bundles

A central concept in this essay is that of families. These aim to capture deformations of objects. We will build these from fibre bundles.

Definitions 0.3. Let E, B and F be complex manifolds, and let $\pi : E \to B$ be a holomorphic surjection. A *(holomorphic) fibre bundle* consists of the data (E, B, F, π) subject to the local triviality condition that there exists an open cover $\{U_\alpha\}$ of B with for each U_α in the open cover a biholomorphism $\varphi_\alpha : \pi^{-1}(U_\alpha) \to F \times U_\alpha$ such that, letting $\mathrm{proj}_2 : F \times U_\alpha \to U_\alpha$ be the projection onto the second factor, the following diagram commutes:



If we only require that E, B and F are differentiable manifolds, π is $\mathcal{C}^\infty$, and that the φ_α are diffeomorphisms, we say that (E, B, F, π) is a *differentiable fibre bundle.*

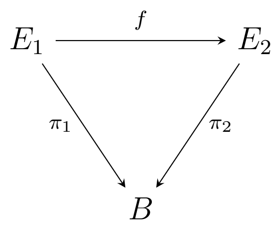
We call E the *total space*, B the *base space*, F the *fibre*, and any collection of pairs $\{(U_\alpha, \varphi_\alpha)\}$ with the above properties a *local trivialisation.* We often refer to E as the fibre bundle, with the rest of the data left implicit. Note that for any $x \in B$ the space $\pi^{-1}(x)$ is biholomorphic to F; we denote this space E_x and call it the *fibre over* x. More generally, for any subset X of B, we call the space $E_X := \pi^{-1}(X)$ the *fibre over* X.

Let G be a group of holomorphic automorphisms of F. We say a fibre bundle with fibres F has structure group G if, given any local trivialisation $\{(U_\alpha, \varphi_\alpha)\}$, the transition functions

$$g_{\alpha\beta}(x) = \varphi_\alpha \varphi_\beta^{-1}\big|_{\{x\}\times F},$$

considered as automorphisms of F, lie in the group G. In this case we refer to G as the *structure group* of the fibre bundle E.

Definition 0.4. Let (E_1, B, F_1, π_1) and (E_2, B, F_2, π_2) be two fibre bundles over the same space B. We say that a holomorphic map $f : E_1 \to E_2$ is a *morphism of fibre bundles*, or a bundle map, if



commutes.

Definition 0.5. Let (E, B, F, π) be a (possibly differentiable) fibre bundle. We call a map $s : B \to E$ a *(global) section* of the fibre bundle if $\pi \circ s = \mathrm{id}_B$. If s is defined only on an open subset $U \subseteq B$, we call s a *local section.*

Example 0.6. Given complex manifolds B and F, the data $(F \times B, B, F, \mathrm{proj}_2)$, where $\mathrm{proj}_2 : F \times B \to B$ is projection onto the second factor, specifies a fibre bundle. We call this the *trivial F-fibred bundle over B.*

Observe that any the projection map π of any differentiable fibre bundle is locally a projection $F \times U \to U$, where F is the fibre. Taking appropriate coordinate neighbourhoods for F and U, and treating them as real manifolds, we see that π is locally the projection $\mathbb{R}^m \to \mathbb{R}^n$ for some natural numbers m and n. We shall call this a submersion.

Definition 0.7. Let $f : X \to Y$ be a smooth map between differentiable manifolds. We call f a *submersion* if the induced tangential map $T_x f : T_x X \to T_{f(x)} Y$ is surjective for all $x \in X$.

Equivalently, f is a submersion if the dimension of X is greater than that of Y and the Jacobian of f at each point of X is of maximal rank.

Definition 0.8. Let $f : X \to Y$ be a continuous map between Hausdorff spaces. We call f *proper* if the preimage of any compact subset of Y is compact.

Observe that, if the fibre F of a fibre bundle is compact, then the projection π is a proper surjective submersion. It is a theorem of Ehresmann that these properties are enough to determine that a map π is the projection of some differentiable fibre bundle.

Theorem 0.9 (Ehresmann fibration theorem)**.** *Let M, B be differentiable manifolds and $f : M \to B$ a proper surjective submersion. Then (M, B, F, f) is a differentiable fibre bundle with fibre $F := f^{-1}(b)$ for any $b \in B$.*

For a proof see Dundas [6, §9.5] or Ebeling [7, §4.3].

Fibre Products

By definition, our maps of bundles are required to be between bundles over a common base space. In order to discuss relationships between bundles over different base spaces, we use the notion of a pullback, or fibre product.

Let X, Y and Y' be complex manifolds, and let $\pi : X \to Y$ and $f : Y' \to Y$ be holomorphic maps. We define the *fibre product* of X and Y' over Y to be the set

$$X \times_Y Y' := \{(x, y) \in X \times Y' \mid f(x) = \pi(y)\}$$

equipped with the subspace topology from $X \times Y'$. This comes equipped with continuous maps

$$\begin{aligned} \widehat{\pi} : X \times_Y Y' &\longrightarrow Y'; \\ (x, y) &\longmapsto y \end{aligned}$$

and

$$\begin{aligned} \widehat{f} : X \times_Y Y' &\longrightarrow X; \\ (x, y) &\longmapsto x. \end{aligned}$$

This gives the following commutative diagram.

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{\widehat{f}} & X \\ {\scriptstyle \widehat{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y, \end{array}$$

It can be shown that the fibre product $X \times_Y Y'$ is in fact universal with respect to this diagram in the category of topological spaces.

Observe that we have set up the notation for fibre products somewhat asymmetrically. This is as fibre products will be of primary interest to us in the case that $\pi : X \to Y$ is a fibre bundle, and $f : Y' \to Y$ is a map of base spaces. From this perspective we shall also call the space $X \times_Y Y'$ the *pullback of X along f* and the map $\widehat{\pi}$ the *pullback of π along f*, and use the notations $f^*X := X \times_Y Y'$ and $f^*\pi := \widehat{\pi}$.

A priori a fibre product is just a topological space, and the induced maps nothing more than continuous. In the cases we are concerned with, however, we have more than this.

Proposition 0.10. *If $\pi : X \to Y$ is a holomorphic surjective submersion, then given any complex manifold Y' and any holomorphic map $f : Y' \to Y$ there exists a unique complex structure on $X \times_Y Y'$ such that it is a complex submanifold of $X \times Y'$ and $\widehat{\pi}$ is a holomorphic surjective submersion.*

In general for the fibre product to again be a complex manifold it suffices that the maps π and f are transverse—that is, given any $(x, y) \in X \times_Y Y'$, that the sum of the images of the tangential maps $(\pi_*)_x$ and $(f_*)_y$ equal the whole of the tangent space $T_{\pi(x)}Y$. Since for a submersion we already have $\operatorname{Im}((f_*)_y) = T_{f(y)}Y$, every map π is transverse to f. See Fritzsche and Grauert [10, Ch.IV §1] for more details.

Note also that a section $s : Y \to X$ of a map $\pi : X \to Y$ pulls back to a section $f^*s : Y' \to f^*X$ of the pullback that maps $y \in Y'$ to $(s \circ f(y), y) \in X \times_Y Y' = f^*X$.

Observe that given the trivial F-fibred bundle $F \times U$ over U and a holomorphic map $f : U' \to U$, the pullback $f^*(F \times U)$ is biholomorphic to the product manifold $F \times U'$. We may thus consider it as the trivial F-fibred bundle over U'. Given a trivialising open cover $\{U_\alpha\}$ of a fibre bundle (E, B, F, π) and a holomorphic map of complex manifolds $f : B' \to B$ then, this shows that the collection of open sets $\{f^{-1}(U_\alpha)\}$ gives a trivialising open cover for the pullback f^*E. We have thus sketched a proof of the following proposition.

Proposition 0.11. *Let (E, B, F, π) be a holomorphic fibre bundle over B, and let $f : B' \to B$ be a holomorphic map. Then $(f^*E, B', F, f^*\pi)$ is a holomorphic fibre bundle over B'.*

Quotients

We shall return a number of times to the question of when a structure descends under a quotient by a group. For now we prove a simple preliminary result in the case of complex manifolds.

Let G be a group acting on a topological space X on the left. Let G have the discrete topology. We say G *acts continuously* if the map

$$\begin{aligned} G \times X &\longrightarrow X; \\ (g, x) &\longmapsto gx \end{aligned}$$

is continuous. We always require this to be true. Furthermore, when X is a complex manifold, we further require that G *acts analytically*; that is, that the above map is holomorphic. This means that each $g \in G$ induces a biholomorphic automorphism of X. We call a group G a complex Lie group if it has a complex structure and composition and inversion are holomorphic maps. In this case we also expect G to act analytically.

We will chiefly consider actions with the following property.

Definition 0.12. The action of a group G on a space X is called *properly discontinuous* if for all $x, y \in X$ there are open neighbourhoods U and V of x and y respectively such that the set

$$\{g \in G \mid gU \cap V \neq \varnothing\}$$

is finite.

As usual, the action of G is called *free* if for all nonidentity $g \in G$ and all $x \in X$ we have $gx \neq x$. We shall show that when a group acting on a complex manifold has both these properties, then the quotient has a natural complex structure of the same dimension.

Lemma 0.13. *Let G be a group acting freely and properly discontinuously on a Hausdorff space X. Then there exist open neighbourhoods U of x and V of y such that $gU \cap V \neq \varnothing$ if and only if $gx = y$.*

Proof. Since G acts properly discontinuously, there exist open neighbourhoods U_0 of x and V_0 of y such that the set

$$I = \{g \in G \mid gU_0 \cap V_0 \neq \varnothing\}$$

is finite, of the form $\{g_1, \ldots, g_n\}$. Let $z_i = g_i x$ for $i = 1, \ldots, n$. These points are distinct as G acts freely. We now consider two cases.

Suppose first that y lies in the G-orbit of x. In this case there exists $g \in G$ such that $gx = y$, unique since G acts freely. Furthermore, $y \in gU_0 \cap V_0$, so g is equal to g_j for some j. Without loss of generality we may suppose $g = g_1$ and hence $y = z_1$. Choose W_i to be pairwise disjoint open neighbourhoods of the z_i; these exist as X is Hausdorff and there are only finitely many z_i. Define

$$U := \bigcap_{i=1}^{n} g_i^{-1}(W_i) \cap U_0, \qquad V := W_1 \cap V_0.$$

Then for $g = g_1$, $y \in g_1 U \cap V$, so $g_1 U \cap V \neq \varnothing$. For $g \in I \setminus \{g_1\}$, $g = g_i$ for some $i \neq 1$, so we have $g_i U \subset W_i$ and $V \subset W_1$, and so $g_i U \cap V = \varnothing$. For $g \notin I$, we have $gU \subset U_0$ and $V \subset V_0$, so $gU \cap V = \varnothing$. This covers all elements of G and hence proves the lemma in the case that x and y lie in the same G-orbit.

Suppose on the other hand that x and y lie in distinct orbits of G. Then there is no $g \in G$ such that $gx = y$ so in particular y is distinct from each z_i. Choose open sets W_i and $V \subset V_0$ to be pairwise disjoint open neighbourhoods of the z_i and of y, respectively. Again these exist as X is Hausdorff and there are only finitely many z_i. We again also define $U = \bigcap_{i=1}^{n} g_i^{-1}(W_i) \cap U_0$. The result now follows by considering the cases $g \in I$ and $g \notin I$ as before. □

Remark 0.14. We may unpack the details of the above lemma to get the following. Let G be a group acting freely and properly discontinuously on a Hausdorff space X. Then:

(i) If $x \in X$, then there exists an open neighbourhood U of x such that $gU \cap U = \varnothing$ for all nonidentity $g \in G$.

(ii) If $x, y \in X$ lie in the same G-orbit, with $g_0 x = y$, then there exist open neighbourhoods U of x and V of y such that $gU \cap V = \varnothing$ except when $g = g_0$.

(iii) If $x, y \in X$ do not lie in the same G-orbit, then there exist open neighbourhoods U of x and V of y such that $gU \cap V = \varnothing$ for all $g \in G$.

Lemma 0.15. *Let G be a group acting freely and properly discontinuously on an n-dimensional complex manifold X. Then there exists a unique n-dimensional complex structure on the quotient $G\backslash X$ such that the quotient projection $\pi : X \to G\backslash X$ is a holomorphic covering map. With this complex structure π is in fact locally biholomorphic.*

Proof. The topological space $G\backslash X$ is connected as it is the quotient of a connected space, and Hausdorff by Remark 0.14(iii) above.

We wish to give a complex structure to $G\backslash X$. Let $x \in G\backslash X$, and choose $\widetilde{x} \in \pi^{-1}(x)$. Remark 0.14(i) gives an open neighbourhood U of $\widetilde{x}$ such that

gU is disjoint from U for all nonidentity $g \in G$. Then $\pi : U \to \pi(U)$ is a homeomorphism. Since X is a complex manifold, there exists a map $z : U \to \mathbb{C}^n$ such that (U, z) forms a coordinate neighbourhood of $\widetilde{x}$. We let $(\pi(U), z \circ \pi^{-1}|_{\pi(U)})$ be a coordinate neighbourhood of x. All such coordinate neighbourhoods on $G\backslash X$ are compatible as they descend from compatible neighbourhoods on X. Since x was arbitrary, these neighbourhoods cover $G\backslash X$, and so form a complex structure.

Under this complex structure it is clear that the quotient map is locally biholomorphic. It is a covering map as for any $x \in G\backslash X$, we may take the open neighbourhood U of a point $\widetilde{x} \in \pi^{-1}(x)$ given by Remark 0.14(i). Then $\pi(U)$ is a neighbourhood of x such that $\pi^{-1}(\pi(U))$ consists of disjoint open sets gU, $g \in G$, each homeomorphic to $\pi(U)$.

To see that this complex structure is the unique structure with this property, observe that if $\pi : X \to G\backslash X$ is a holomorphic covering a function $f : G\backslash X \to \mathbb{C}$ is holomorphic if and only if $f \circ \pi$ is holomorphic. Since the sheaf of holomorphic functions on a complex manifold uniquely determines the complex structure, the structure is unique. □

Differential Forms

The theory of integration on Riemann surfaces will be our key tool in relating elliptic curves with lattices. While it is not too much more effort to develop a theory of integration for all complex manifolds, for simplicity's sake we shall now restrict our attention to Riemann surfaces.

Definition 0.16. Let X be a Riemann surface with atlas $\{(U_i, z_i)\}_{i\in I}$. A *differential 1-form* ω on X is a collection of differential 1-forms $\omega_i = f_i(z_i, \overline{z}_i)dz_i + g_i(z_i, \overline{z}_i)d\overline{z}_i$ on the open subsets U_i of $\mathbb{C}$, where f_i, g_i are smooth functions, such that on the overlaps $U_i \cap U_j$ we have

$$f_j(z_j, \overline{z}_j) = f_i(w_{ij}(z_j), \overline{w_{ij}(z_j)})\frac{dw_{ij}(z_j)}{dz_j}$$

and

$$g_j(z_j, \overline{z}_j) = g_i(w_{ij}(z_j), \overline{w_{ij}(z_j)})\overline{\frac{dw_{ij}(z_j)}{dz_j}},$$

where $w_{ij} = z_i \circ z_j^{-1}$ (and thus $w_{ij}(z_j) = z_i$) as above.

A *holomorphic 1-form* is a differential 1-form for which each f_i is holomorphic and each g_i zero. We denote the space of holomorphic 1-forms on X by $H^0(X, \Omega^1_X)$. Similarly, a *meromorphic 1-form* is a differential 1-form for which each f_i is meromorphic and each g_i zero.

Integration and differentiation of differential forms are as in the case of real manifolds. Note that all holomorphic 1-forms are closed: If $\omega = f dz$ is a holomorphic 1-form, then

$$d\omega = (\tfrac{\partial f}{\partial z} dz + \tfrac{\partial f}{\partial \bar{z}} d\bar{z}) \wedge dz = \tfrac{\partial f}{\partial z} dz \wedge dz = 0,$$

since the holomorphicity of f implies $\frac{\partial f}{\partial \bar{z}} = 0$.

The Genus of a Compact Riemann Surface

It is well-known that a compact orientable 2-real-dimensional manifold X is homeomorphic to a sphere with g-handles, for some nonnegative integer g. We call this number the *genus* of X. As the complex plane has a natural orientation, and holomorphic functions preserve this orientation, every Riemann surface is orientable. In particular, the genus $g(X)$ of a compact Riemann surface X is well-defined.

We might call the above genus the 'topological genus' of X, and define the 'analytic genus' to be the dimension of the space $H^0(X, \Omega^1_X)$ of holomorphic 1-forms. It is a fundamental result in the theory of Riemann surfaces that these two definitions of genus agree.

Proposition 0.17.

$$g(X) = \dim H^0(X, \Omega^1_X)$$

For a proof, see Griffiths [11, Theorem 2.1] or Farkas and Kra [8, Proposition III.2.7]. The former makes use of the correspondence between compact Riemann surfaces and algebraic plane curves, while the latter proves the result by discussing harmonic differentials.

The genus is an important topological invariant, and somewhat curiously also carries much information about the analytic structure on a compact Riemann surface. One example of this, which we shall later use, is the following. On any compact Riemann surface the degree of a nonconstant meromorphic function—that is, the difference between its number of zeroes and poles, counting multiplicities—is zero. On the other hand, the degree of a meromorphic 1-form is governed by the genus.

Let ω be a meromorphic 1-form on a compact Riemann surface X, let $p \in X$, and suppose that $\omega = f_i dz_i$ for some coordinate neighbourhood (U_i, z_i) containing the point p. We then set $\mathrm{ord}_p(\omega) = m, -m$ or 0 according to whether f_i has at p a zero of order m, a pole of order m, or neither a zero nor a pole. This is well-defined as the transition functions w_{ij} and their derivatives have no zeroes nor poles. We let $\deg(\omega) = \sum_{p \in X} \operatorname{ord}_p(\omega)$.

For meromorphic 1-forms on a compact Riemann surface, we can then state the Poincaré-Hopf index formula as follows.

Proposition 0.18 (Poincaré-Hopf formula)**.** *Let X be a compact Riemann surface, and let ω be a meromorphic 1-form on X. Then*

$$\deg(\omega) = 2g(X) - 2.$$

This proposition follows directly from the from the Poincaré-Hopf formula for real differential forms and the observation that $\operatorname{ord}_p(\omega) = -\operatorname{Ind}_p(\operatorname{Re}\omega)$. A proof of the real case can be found in Milnor [25, §6].

0.2 Elliptic Curves and Their Families

It is a theorem of Riemann that, up to biholomorphism, the only compact Riemann surface of genus 0 is the Riemann sphere. In this essay we move on to what is in some sense the next problem: compact Riemann surfaces of genus 1. These are, essentially, the elliptic curves. The definitions here follow those of Hain [14] and Kodaira [20].

Definition 0.19. An *elliptic curve* $(E; O)$ is a compact Riemann surface E of genus 1 with a marked point $O \in E$.

We wish that our maps of these objects preserve both the complex structure and the marked point.

Definition 0.20. Given elliptic curves $(E_1; O_1)$ and $(E_2; O_2)$, a *morphism of elliptic curves* is a holomorphic map $f : E_1 \to E_2$ such that $f(O_1) = O_2$.

The exact definition of an elliptic curve in the literature varies quite widely, although numerous alternative definitions are equivalent. By elliptic curve in this document we refer only to what are often called elliptic curves over the complex numbers $\mathbb{C}$, even though elliptic curves may be defined more generally over any field. In this more general case the definition is necessarily approached through more general algebro-geometric means; in general elliptic curves do not form complex manifolds. Many definitions do not include the marked point. Our reasons for including it will become clearer in later chapters, but the moral is it makes the moduli problem more tractable.

Example 0.21. Let Λ be an additive subgroup of $\mathbb{C}$ generated by two nonzero complex numbers not real multiples of each other. (This is essentially the data we shall formalise as a lattice in later chapters.) Let $0 \in \mathbb{C}/\Lambda$ is the image of $0 \in \mathbb{C}$ under the quotient map. Then $(\mathbb{C}/\Lambda, 0)$ is an elliptic curve.

The quotient $\mathbb{C}/\Lambda$ has a natural complex structure as Λ acts freely and properly discontinuously on $\mathbb{C}$. More explicitly, we may note that Λ is a discrete subgroup of $\mathbb{C}$, and hence that the preimage of any simply-connected open set $U \subset \mathbb{C}/\Lambda$ under the quotient map q consists of disjoint open subsets of $\mathbb{C}$ each homeomorphic to U. Taking any connected component V in the preimage, $(U, q|_V^{-1})$ gives a coordinate neighbourhood on $\mathbb{C}/\Lambda$. As any pair of connected component of the preimage of U are translates of one another, any pair of coordinate neighbourhoods constructed in this way is compatible. Thus such neighbourhoods give a complex structure on $\mathbb{C}/\Lambda$. Furthermore, the quotient is topologically a torus, and thus of genus 1. This shows that $(\mathbb{C}/\Lambda, 0)$ is an elliptic curve.

Given a class of geometric objects, by a family of these objects we mean to imply a parametrised collection of these objects such that the geometric structure on these objects in some sense varies consistently with the parameter. For example, given a parametrised collection of surfaces in 3-dimensional Euclidean space, we might give the parameter space a topology and ask that the surfaces vary continuously as a function of the parameter. As we have presented them, however, elliptic curves are not naturally embedded in any space, so we cannot discuss families in this way.

As another example, observe that we call a function $f : B \to \mathbb{C}^n$, where B is an open subset of $\mathbb{C}^m$, a holomorphic function if and only if its graph

$$\{(f(t), t) \in B \times \mathbb{C}^n \mid t \in B\}$$

is a complex submanifold of $\mathbb{C}^{m+n}$. We might think of this function as describing a holomorphic family of points in $\mathbb{C}^n$ parametrised by the base space B, and interpret this requirement on the graph as the criterion that for a family to be holomorphic we require a natural complex structure on the set of points $\{f(t)\}_{t \in B}$. We use this idea to define a family of elliptic curves.

Definition 0.22. A *(holomorphic) family of elliptic curves* consists of the data $(\mathcal{E}, B, \pi, s)$, where $\mathcal{E}$ and B are complex manifolds, $\pi : \mathcal{E} \to B$ is a surjective holomorphic submersion, and $s : B \to \mathcal{E}$ is a holomorphic section of π such that for each $x \in B$ the fibre $(\pi^{-1}(x), s(x))$ over x is an elliptic curve.

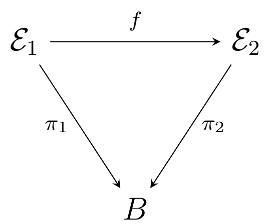
We call $\mathcal{E}$ the *total space*, B the *base space*, and for any subset X of B we call the preimage $E_X := \pi^{-1}(X)$ the *fibre over* X.

We think of this definition as requiring the complex structure and the marked point of a collection of elliptic curves to vary 'holomorphically' with respect to a parameter. In the case that one wishes to consider less rigid deformations of the complex structure, one could equally well consider what might be called 'continuous' families of elliptic curves by specifying that B is only an 'almost complex' manifold, and the maps involved are continuous. For the sake of a clearer exposition we avoid these details; the interested reader might consult [20, Ch.4].

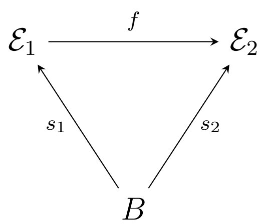
Given an elliptic curve $(E; O)$, we call an elliptic curve $(E'; O')$ a *deformation* of $(E; O)$ if there exists a connected family of elliptic curves $(\mathcal{E}, B, \pi, s)$ and elements x, x' of B such that $(E; O) \cong (\mathcal{E}_x; s(x))$ and $(E'; O') \cong (\mathcal{E}_{x'}; s(x'))$. It can be shown, in a manner similar to that of the proofs of Ehresmann's theorem referenced above, that given a family of elliptic curves $(\mathcal{E}, B, \pi, s)$, the triple $(\mathcal{E}, B, \pi)$ gives a differentiable fibre bundle or, equivalently (by Ehresmann's theorem), that the map π must be proper [20, §2.3]. This shows that the differentiable structure of elliptic curves does not change under deformation. We shall show that there exists a family whose fibres, up to isomorphism, are all elliptic curves, and hence that all elliptic curves are diffeomorphic.

As we wish to classify such families, we should define when we consider two families the same. The first criterion is that both families should lie over the same base space. The second criterion is that, given a point in the base space, then the elliptic curve lying over this point in the first family should be isomorphic to elliptic curve lying over this same point in the second. This alone, however, is not enough: we still need to ensure that the elliptic curves vary locally in the same way. This is captured by the following definition.

Definition 0.23. Let B be complex manifold. A *morphism between families* $(\mathcal{E}_1, B, \pi_1, s_1)$ *and* $(\mathcal{E}_2, B, \pi_2, s_2)$ *of elliptic curves over* B is a holomorphic map $f : \mathcal{E}_1 \to \mathcal{E}_2$ such that



and



commute.

If f is a biholomorphism, then we call f an *isomorphism* and the families $(\mathcal{E}_1, B, \pi_1, s_1)$ and $(\mathcal{E}_2, B, \pi_2, s_2)$ *isomorphic*.

These families will be our main objects of study.

Chapter 1

Moduli Spaces

Our aim is to understand families of elliptic curves. A reasonable place to start is to attempt to classify all such families. A moduli space does just this. In this chapter we put aside elliptic curves for a moment and discuss this concept in general. We shall be, admittedly, somewhat vague in defining a moduli problem, the solution to which we shall call a moduli space. In defence of this approach, I point out that the purpose here is only to convey the spirit of a line of inquiry. Rest assured that we will, in time, get to a precise statement of the moduli problem for elliptic curves, and a rigourous discussion of its solution (or lack thereof).

Here we set up moduli problems as in Harris and Morrison [16]. A slightly different approach is taken in Bruin [3]. More detail on representing functors can be found in Mac Lane [22, Ch.III].

1.1 Moduli Problems

A moduli problem is one of classification. For this we need objects to classify, and a notion of when objects are the same. What sets a moduli problem apart, however, is that the classification is required to be geometric in nature. For this to make sense, we need a third ingredient: an idea of modulation, or how the objects can vary. The solution to a moduli problem then, known as a moduli space, is a space parametrising the isomorphism classes of our objects with a geometry reflecting the ways in which the objects may vary. More precisely, we capture the modulations of our objects by discussing families of them, in the sense introduced in the previous chapter. A moduli space then yields a 'universal' family, cataloguing all objects and the ways they modulate.

Let **Spaces** be the category of possible parameter spaces, which we shall call *base spaces*, for families. Given a base space B, we can construct the set $S(B)$ of isomorphism classes of families of objects over the base space. Furthermore, given a family E over some base space B_2 and any morphism of $f : B_1 \to B_2$ of base spaces, we can construct a set of objects over B_1 by assigning to a point $x \in B_1$ the object in the family E parametrised by $f(x) \in B_2$. For our moduli problem to be well-defined, we require that this in fact constructs a family, and that the isomorphism class of the constructed family depends only on the isomorphism class of the initial family E. This is not an unreasonable requirement: it amounts to little more than expecting that morphisms of base spaces preserve the way objects can modulate. We shall call the family constructed the *pullback family of E by f*. Sending an isomorphism class of families to the isomorphism class of the pullback of a representative by f then gives a map $S(f) : S(B_2) \to S(B_1)$. We thus have a contravariant functor

$$S : \mathbf{Spaces} \longrightarrow \mathbf{Set}.$$

We call this functor the *moduli functor*.

For any base space M we can define a contravariant functor $\mathrm{Mor}(\,\cdot\,, M) : \mathbf{Spaces} \to \mathbf{Set}$ taking any base space B to the set $\mathrm{Mor}(B, M)$ of morphisms from B to M, and any morphism $f : B_1 \to B_2$ of base spaces to the map

$$\begin{aligned} f^* : \mathrm{Mor}(B_2, M) &\longrightarrow \mathrm{Mor}(B_1, M) \\ g &\longmapsto g \circ f. \end{aligned}$$

We call this functor the *functor of points of M*.

A moduli problem then consists of finding a base space M such that there exists a natural isomorphism between the moduli functor and the functor of points of M. If such an M exists we call M the *moduli space* of our moduli problem and say that the moduli functor is *representable* by M. It can be shown, using Yoneda's lemma, that a moduli space is unique up to unique isomorphism.

To be clear, although a moduli problem is motivated by choosing objects, their equivalences and their permissible modulations, we more rigourously define one only by specifying the elements of a moduli functor: a category of base spaces, a set of equivalence classes of families and a notion of pullback for families.

As an example, we give the central problem of this essay: the moduli problem for elliptic curves. Define the functor

$$\mathscr{E}\!\ell\ell : \mathbf{Mfld}_{\mathbb{C}} \longrightarrow \mathbf{Set}$$

such that for any complex manifold B we have

$$\mathscr{E}\!\ell\ell(B) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{families of elliptic curves over } B \end{array} \right\},$$

and such that for any holomorphic map of complex manifolds $f : B_1 \to B_2$ the function $\mathscr{E}\!\ell\ell(f) : \mathscr{E}\!\ell\ell(B_2) \to \mathscr{E}\!\ell\ell(B_1)$ maps the isomorphism class of a family of elliptic curves over B_2 to the isomorphism class of its pullback along f. It is an immediate consequence of Proposition 0.11 that the pullback of a family of elliptic curves is again a family of elliptic curves, and it is readily seen that up to isomorphism the pullback family only depends on the isomorphism class of the given family. Thus this functor is well-defined. The moduli problem for elliptic curves then asks for a complex manifold M that represents the functor $\mathscr{E}\!\ell\ell$.

We shall spend the rest of this essay discussing this problem.

The Universal Family

The moduli space, if it exists, is a very useful tool for studying objects and their families. For example, the moduli space directly gives a classification of all objects up to isomorphism. Observe that a family of objects over a single point is simply a single object. Thus the isomorphism classes of families over a point are in one-to-one correspondence with the isomorphism classes of objects. But, by the definition of the moduli space, the isomorphism classes of families over a point are in one-to-one correspondence with maps from the point into the moduli space. In all cases we are concerned with, every point in the moduli space can be realised as the image of such a map. Thus we have the general principle:

> The points of a moduli space are in one-to-one correspondence with the isomorphism classes of objects of the moduli problem.

We note that when dealing with schemes subtleties come into play regarding the definition of a point, and stress that the above is only a guiding principle. It nonetheless suffices for our discussions in this essay.

A stronger consequence is the following. Given any family E over any base space B, the natural isomorphism returns a morphism of $f_E : B \to M$. This morphism sends a point $x \in B$ to the unique point $f_E(x)$ of M such that the objects over x and $f_E(x)$ are isomorphic. Observe also that any moduli space comes equipped with, up to isomorphism, a canonical family $\mathcal{U}$. This family is that corresponding to the identity morphism $1_M \in \operatorname{Mor}(M, M)$ under the natural isomorphism between the moduli functor and the functor of points of M. Since $f_E = f_E^*(1_M)$, the commutativity of the naturality square

$$\begin{array}{ccc} S(M) & \xrightarrow{S(f_E)} & S(B) \\ \updownarrow & & \updownarrow \\ \mathrm{Mor}(M, M) & \xrightarrow[f_E^*]{} & \mathrm{Mor}(B, M) \end{array}$$

thus shows that the family E is isomorphic the pullback of $\mathcal{U}$ along f_E and, moreover, that it is isomorphic in a natural way. This shows that:

> Over any moduli space there exists a family from which every family arises as a pullback in a unique way.

For this reason we call this family $\mathcal{U}$ the *universal family*. Here the benefits of a moduli space become obvious: not only are the families over a given base space in bijection with maps from this base space into the moduli space, but these families all arise as pullbacks. Thus by studying the universal family alone, we can deduce facts about *all* families.

Conversely, suppose that we have a universal family over some space M. For each base space B, we then have a bijection $\tau_B : \mathrm{Mor}(B, M) \to S(B)$ that takes a morphism to the pullback of the universal family along it. This then defines a natural isomorphism between the functors S and $\mathrm{Mor}(\,\cdot\,, M)$—by the definition of a universal family we require that the naturality squares commute. Thus to solve a moduli problem it is necessary and sufficient to find a space with a universal family over it.

Unfortunately, not all moduli problems can be solved. In the next section we consider an example of one such problem.

1.2 Lattices

This section discusses what we will call the moduli problem for lattices. Lattices, it will turn out, are closely related to elliptic curves, and we shall find ourselves working with them in the chapters to come. For this reason we go through this example in some depth, perhaps at the expense of a little conciseness.

For our purposes, a lattice is an evenly-spaced grid of points in the plane, and two lattices are equivalent if we can scale and rotate one to form the other. We formalise this as follows.

Definitions 1.1. A *lattice* (V, Λ, φ) consists of a 1-dimensional complex vector space V, a rank two free abelian group Λ, and a group homomorphism $\varphi : \Lambda \hookrightarrow V$ such that the image $\varphi(\Lambda)$ spans V as a real vector space.

Given two lattices $(V_1, \Lambda_1, \varphi_1)$ and $(V_2, \Lambda_2, \varphi_2)$, a *morphism of lattices* (f_V, f_Λ) is a complex linear map $f_V : V_1 \to V_2$ and a group homomorphism $f_\Lambda : \Lambda_1 \to \Lambda_2$ such that

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\varphi_1} & V_1 \\ {\scriptstyle f_\Lambda}\downarrow & & \downarrow{\scriptstyle f_V} \\ \Lambda_2 & \xrightarrow{\varphi_2} & V_2 \end{array}$$

commutes.

Isomorphic lattices are often also called *homothetic*. Note that the group homomorphism φ defining a lattice must be injective, and thus is an isomorphism onto its image in V. Where no confusion will arise, we shall also refer to the subgroup of V given by the image $\varphi(\Lambda)$ of Λ as a lattice. We picture this as follows.

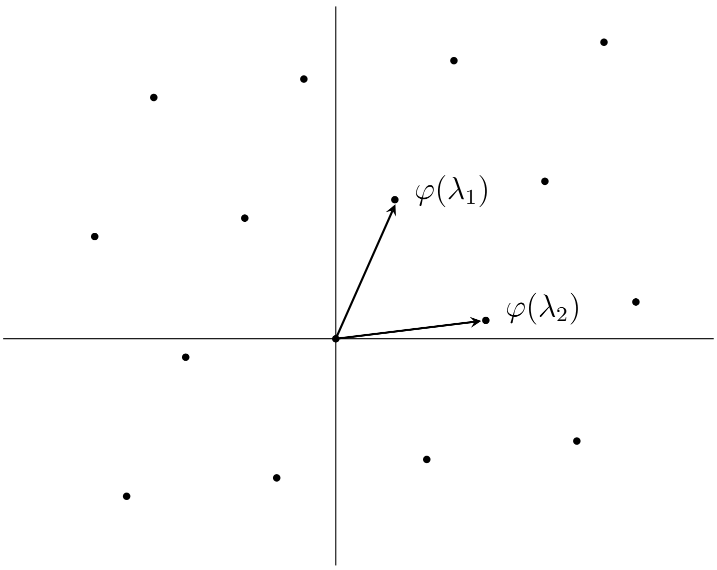


Figure 1.1: Choosing an isomorphism with $\mathbb{C}$, we may picture the 1-dimensional complex vector space V as a plane. The points in the image of φ then give an evenly spaced grid. It is in this sense that (V, Λ, φ) is a lattice. Here λ_1, λ_2 represent a basis for Λ.

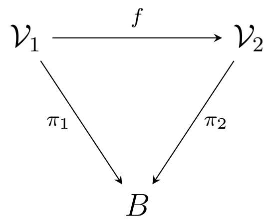
Suppose we are given a space B indexing a set of lattices $\{(V_x, \Lambda_x, \varphi_x)\}_{x \in B}$. If we identify all complex vector spaces V_x with $\mathbb{C}$ and free abelian groups Λ_x

with $\mathbb{Z}^2$, the notion of a continuous or smooth or analytic family is intuitive: for each element α in the group $\mathbb{Z}^2$ the point $\varphi_x(\alpha) \in \mathbb{C}$ should vary continuously or smoothly or analytically as a function of x. This defines a family of lattices over B that lies in the what we might call a trivial family of vector spaces, $\mathbb{C} \times B$. In general a family may lie in any family of vector spaces, not just the trivial one. Thus in order to define a family of lattices, we start by specifying what we mean by a family of vector spaces. What we mean by this is a vector bundle.

Definitions 1.2. Let $\mathcal{V}$ and B be complex manifolds, let $\pi : \mathcal{V} \to B$ be a holomorphic surjection, and let n be a positive integer. A *(holomorphic) vector bundle of dimension* n consists of the data $(\mathcal{V}, B, \pi)$ subject to the condition that for each $x \in B$ the fibre $\mathcal{V}_x := \pi^{-1}(\{x\})$ over x has the structure of an n-dimensional complex vector space, and also subject to the local triviality condition that there exists an open cover of B by open sets U_α for each of which there exists a biholomorphism $\varphi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{C}^n \times U_\alpha$ that takes each fibre $\mathcal{V}_x$ over a point $x \in U_\alpha$ to the vector space $\mathbb{C}^n \times \{x\}$ by a complex linear isomorphism.

As for fibre bundles, we call $\mathcal{V}$ the *total space*, B the *base space*, and any collection of pairs $\{(U_\alpha, \varphi_\alpha)\}$ with the above properties a *local trivialisation.*

Fixing a base space B, we define a *morphism of vector bundles over* B between two vector bundles $(\mathcal{V}_1, B, \pi_1)$ and $(\mathcal{V}_2, B, \pi_2)$ to be a holomorphic map $f : \mathcal{V}_1 \to \mathcal{V}_2$ such that



commutes, and such that for every $x \in B$ the induced map $f|_{(\mathcal{V}_1)_x} : (\mathcal{V}_1)_x \to (\mathcal{V}_2)_x$ is a complex linear map of vector spaces.

Observe that a vector bundle is precisely a fibre bundle with fibres $\mathbb{C}^n$ and structure group $\mathrm{GL}(n, \mathbb{C})$. We are particularly interested in vector bundles of dimension 1. We call these *line bundles.*

Example 1.3. For any topological space B, the product space $\mathbb{C}^n \times B$ is a vector bundle over B, with a local trivialisation simply consisting of the whole base space B with the trivial isomorphism of $\mathbb{C}^n \times B$ with itself. We call this the *trivial vector bundle of dimension* n *over* B. The line bundle $\mathbb{C} \times B$ is the *trivial line bundle over* B. Note that an n-dimensional vector bundle is trivial if and only if there exists n global sections whose images are linearly independent on each fibre.

Perhaps a more interesting example is the following.

Example 1.4 (The holomorphic cotangent bundle). Let X be a Riemann surface, and $\{(U_i, z_i)\}$ a coordinate covering of X. On each open set U_i, which is biholomorphic to an open set in $\mathbb{C}$, there exists the trivial cotangent bundle $\mathbb{C} \times U_i$. We define the *holomorphic cotangent bundle*, or *canonical line bundle*, of X to be the line bundle given by gluing these trivial bundles together using the transition functions $g_{ij} = dz_j/dz_i$ on each intersection $U_i \cap U_j$. Observe that the fibres of this bundle over each point of X are naturally isomorphic to the cotangent space at the point.

A vector bundle is a fibre bundle with some extra structure on each fibre. In particular, each fibre is given the structure of a group. A vector bundle is thus an example of what we shall call a group bundle. These are the group objects in the category of fibre bundles.

Definition 1.5. Let B be a complex manifold. A *group bundle* $(E, B, F, \pi, \mu, e, i)$ *over* B is a fibre bundle (E, B, F, π) together with bundle maps

$$\mu : E \times_B E \longrightarrow E, \qquad e : B \longrightarrow E, \qquad i : E \longrightarrow E$$

such that the following three diagrams commute:

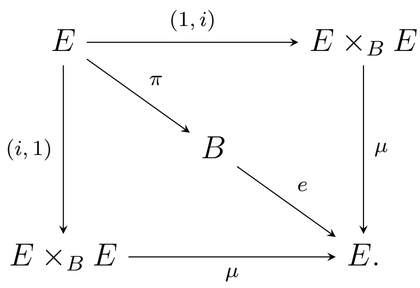
(i)

$$\begin{CD} E \times_B E \times_B E @>{1 \times \mu}>> E \times_B E \\ @V{\mu \times 1}VV @VV{\mu}V \\ E \times_B E @>>{\mu}> E \end{CD}$$

(ii)

$$\begin{CD} B \times_B E \times_B B @>{e \times 1 \times 1}>> E \times_B E \times_B B \cong E \times_B E \\ @V{1 \times 1 \times e}VV @VV{\mu}V \\ B \times_B E \times_B E \cong E \times_B E @>>{\mu}> E \end{CD}$$

(iii)



Example 1.6. Given a complex Lie group G and a complex manifold X, the *trivial G-fibred bundle over X* is the space $G \times X$ equipped with the maps

$$\begin{aligned} \mu : (G \times X) \times_X (G \times X) &\longrightarrow G \times X \\ ((g, x), (h, x)) &\longmapsto (gh, x), \end{aligned}$$

$$\begin{aligned} e : X &\longrightarrow G \times X \\ x &\longmapsto (e, x), \end{aligned}$$

$$\begin{aligned} i : G \times X &\longrightarrow G \times X \\ (g, x) &\longmapsto (g^{-1}, x), \end{aligned}$$

inherited from the group maps on G. This is by inspection a group bundle.

Just as a lattice consists of the group Λ lying inside a vector space isomorphic to $\mathbb{C}$, we wish to define a family of lattices as a group bundle with fibres isomorphic to Λ lying inside a vector bundle with fibres isomorphic to $\mathbb{C}$. We shall call the former bundle a Λ-fibred bundle. In general, we wish to say that a G-fibred bundle is a group bundle locally isomorphic (as a group bundle) to the trivial G-fibred bundle $G \times X$. We specify what we mean by a morphism of group bundles.

Definition 1.7. Let $(E_1, B, G_1, \pi_1, \mu_1, e_1, i_1)$ and $(E_2, B, G_2, \pi_2, \mu_2, e_2, i_2)$ be group bundles over a common base space B. Then a bundle map $f : E_1 \to E_2$ over B is a *group bundle map (over B)* if the following diagram commutes:

$$\begin{CD} E_1 \times_B E_1 @>{\mu_1}>> E_1 \\ @V{f \times f}VV @VV{f}V \\ E_2 \times_B E_2 @>>{\mu_2}> E_2. \end{CD}$$

Considering things fibrewise shows that the following two diagrams also must commute:

$$\begin{array}{ccc} & & E_1 \\ & \nearrow^{e_1} & \big\downarrow f \\ B & & \\ & \searrow_{e_2} & \\ & & E_2, \end{array}$$

$$\begin{array}{ccc} E_1 & \xrightarrow{i_1} & E_1 \\ f\big\downarrow & & \big\downarrow f \\ E_2 & \xrightarrow[i_2]{} & E_2. \end{array}$$

Together these commutative diagrams specify that the group structure on E_1 is carried over to the group structure on E_2 by the bundle map. Equivalently, we might define a group bundle map as a bundle map that is a group homomorphism on each fibre. As per usual, a group bundle map f is a group bundle isomorphism if there exists a group bundle map g that is a left and right inverse to f.

Definition 1.8. Let G be a complex Lie group. A group bundle $(E, B, F, \pi, \mu, e, i)$ is a G-fibred bundle if there exists an open cover $\{U_\alpha\}$ of B such that for each U_α in the open cover there exists a group bundle isomorphism $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G \times U_\alpha$ over U_α.

Note that each fibre $E_x \cong F$, considered as a group by restricting the maps μ, e and i to this fibre, is isomorphic as a complex Lie group to G. Since the transition functions on each fibre are also group homomorphisms, they must be group automorphisms. This implies that the structure group of a G-fibred bundle is $\mathrm{Aut}(G)$. As a discrete group, consider the group Λ a complex Lie group of dimension 0. In the case of a Λ-fibred bundle, the above implies the transition functions must lie in $GL(2, \mathbb{Z})$, the group of automorphisms of Λ.

We are now, finally, in a position to define a family of lattices. One might expect that we should consider continuous families of lattices; after all, we have so far not come across any reason to impose stricter conditions on the way lattices modulate. We will, however, concern ourselves with analytic families instead. Although lattices themselves have little analytic structure to worry about, we wish later to relate families of lattices with those of elliptic curves.

Observe that over any complex manifold a vector bundle is locally the product of two complex manifolds, and Λ-fibred bundle is locally the disjoint union of complex manifolds. In both cases this gives a complex structure.

Definition 1.9. Let Λ be the free abelian group on two generators. A *(holomorphic) family of lattices* $(\mathcal{V}, \mathcal{L}, B, \Phi)$ over a complex manifold B consists of a 1-dimensional vector bundle $\mathcal{V}$ over B, a Λ-fibred bundle $\mathcal{L}$ over B, and a holomorphic group bundle morphism $\Phi : \mathcal{L} \to \mathcal{V}$ with the property that for each $x \in B$, the triple $(\mathcal{V}_x, \mathcal{L}_x, \Phi|_{\mathcal{L}_x})$ is a lattice.

The definition of a map of families of lattices is analogous to that of a map of lattices.

Definition 1.10. Given two families of lattices $(\mathcal{V}_1, \mathcal{L}_1, B, \Phi_1)$ and $(\mathcal{V}_2, \mathcal{L}_2, B, \Phi_2)$, a *morphism of families of lattices* (F_V, F_Λ) is a holomorphic vector bundle map $F_V : \mathcal{V}_1 \to \mathcal{V}_2$ together with a holomorphic group bundle map $F_\Lambda : \mathcal{L}_1 \to \mathcal{L}_2$ such that

$$\begin{array}{ccc} \mathcal{L}_1 & \xrightarrow{\Phi_1} & \mathcal{V}_1 \\ {\scriptstyle F_\Lambda}\downarrow & & \downarrow{\scriptstyle F_V} \\ \mathcal{L}_2 & \xrightarrow{\Phi_2} & \mathcal{V}_2 \end{array}$$

commutes.

Lastly, we briefly discuss pullbacks of families of lattices. Let $(\mathcal{V}, \mathcal{L}, B, \Phi)$ be a family of lattices. Observe that the projection maps of the line bundle $\pi_{\mathcal{V}} : \mathcal{V} \to B$ and the Λ-fibred bundle $\pi_{\mathcal{L}} : \mathcal{L} \to B$ are necessarily submersions, so given a complex manifold B' and a holomorphic map $f : B' \to B$ the pullbacks $f^*\pi_{\mathcal{V}} : f^*\mathcal{V} \to B'$ and $f^*\pi_{\mathcal{L}} : f^*\mathcal{L} \to B'$ give bundles of the same type. Define the map

$$f^*\Phi : f^*\mathcal{L} = \mathcal{L} \times_B B' \longrightarrow f^*\mathcal{V} = \mathcal{V} \times_B B';$$
$$(a, b) \longmapsto (\Phi(a), b).$$

Since $(\mathcal{V}, \mathcal{L}, B, \Phi)$ is a family of lattices, $\pi_{\mathcal{V}}(\Phi(a)) = \pi_{\mathcal{L}}(a)$, and so this map is well-defined. It is straightforward to check that $f^*\Phi$ is a group bundle map, and that the restrictions of $f^*\mathcal{V}$, $f^*\mathcal{L}$ and $f^*\Phi$ to the fibres over each point in B' does give a lattice. This shows that $(f^*\mathcal{V}, f^*\mathcal{L}, B', f^*\Phi)$ is a family of lattices, and hence that the pulling back a family of lattices as fibre bundles again gives a family of lattices.

Now that we have defined lattices, their families and pullbacks of their families, we turn to the question of their moduli.

1.3 An Unsolvable Moduli Problem

The lesson of this section is that the moduli problem for lattices is unsolvable, and that the reason for this is the existence of nontrivial automorphisms of lattices.

Define the functor

$$\mathscr{Latt} : \mathbf{Mfld}_{\mathbb{C}} \longrightarrow \mathbf{Set}$$

such that for any complex manifold B we have

$$\mathscr{Latt}(B) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{families of lattices over } B \end{array} \right\}$$

and such that for any holomorphic map of complex manifolds $f : B_1 \to B_2$ the function $\mathscr{Latt}(f) : \mathscr{Latt}(B_2) \to \mathscr{Latt}(B_1)$ maps the isomorphism class of a family of lattices over B_2 to the isomorphism class of its pullback along f. We have seen that pullbacks exist in the category of holomorphic group bundles, and so this is well-defined.

Suppose that a moduli space for lattices exists. This implies we have a space M, above which lies a universal family. Suppose also that we have a family of lattices over B, and that all fibres over B are isomorphic. We call such a family *isotrivial*. Then, when mapping this family into our moduli space, B must map to a single point, and hence the pullback family on B is isomorphic to the trivial family. But the pullback family must also be isomorphic to the original family over B. Thus, if a moduli space exists, then all isotrivial families over B must be trivial. To prove no moduli space exists it hence suffices to construct a nontrivial isotrivial family over some space B.

Example 1.11 (A nontrivial isotrivial family of lattices). We shall construct such a family above the base space $B = \mathbb{C}^\times$, the complex plane minus the origin. Our line bundle $\mathcal{V}$ shall be the trivial line bundle $\mathbb{C} \times \mathbb{C}^\times$. The construction of our Λ-fibred bundle $\mathcal{L}$ is a bit more complicated. We use the two charts $U = \mathbb{C} \setminus [0, \infty)$ and $V = \mathbb{C} \setminus (-\infty, 0]$, where the two intervals are those in the real line. Then define $\mathcal{L}$ to be the Λ-fibred bundle over $\mathbb{C}$ given by patching together the two trivial bundles $\mathbb{Z}^2 \times U$ and $\mathbb{Z}^2 \times V$ using the transition function

$$t_{U,V} : \mathbb{Z}^2 \times (U \cap V) \longrightarrow \mathbb{Z}^2 \times (U \cap V);$$
$$\big((m,n),z\big) \longmapsto \begin{cases} \big((m,n),z\big) & \text{if } 0 < \arg z < \pi, \\ \big((-m,-n),z\big) & \text{if } \pi < \arg z < 2\pi. \end{cases}$$

Let now $z \mapsto \sqrt{z}$ be the function defined on U that is the inverse to the map $z \mapsto z^2$ when restricted to $0 < \arg z < \pi$, and let $z \mapsto \sqrt{z}^*$ be the function defined on V that is inverse to the map $z \mapsto z^2$ when restricted to $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$. Note that both of these maps are holomorphic. We then define a group bundle map $\Phi : \mathcal{L} \to \mathcal{V}$ by setting

$$\begin{aligned} \Phi_U : \mathbb{Z}^2 \times U &\longrightarrow \mathbb{C} \times U; \\ \big((m,n), z\big) &\longmapsto \big((m+in)\sqrt{z}, z\big), \end{aligned}$$

and

$$\begin{aligned} \Phi_V : \mathbb{Z}^2 \times V &\longrightarrow \mathbb{C} \times V; \\ \big((m,n), z\big) &\longmapsto \big((m+in)\sqrt{z}^*, z\big). \end{aligned}$$

Since

$$\sqrt{z} = \begin{cases} \sqrt{z}^* & \text{if } 0 < \arg z < \pi, \\ -\sqrt{z}^* & \text{if } \pi < \arg z < 2\pi, \end{cases}$$

the diagram

$$\begin{array}{ccc} \mathbb{Z}^2 \times (U \cap V) & \xrightarrow{\Phi_U} & \\ {\scriptstyle t_{U,V}} \big\downarrow & & \mathbb{C} \times (U \cap V) \\ \mathbb{Z}^2 \times (U \cap V), & \xrightarrow{\Phi_V} & \end{array}$$

commutes, and so Φ is well-defined. By inspection we see that the image of $\big((1,0), z\big)$ and $\big((0,1), z\big)$ are orthogonal in $\mathbb{C}$, and hence $\mathbb{R}$-linearly independent. Thus $(\mathcal{V}, \mathcal{L}, \mathbb{C}^\times, \Phi)$ is a family of lattices. One way of thinking of this family is as an analogue of a Möbius strip: moving around the origin twists the lattice a half-rotation.

It remains to show that we have defined a nontrivial isotrivial family. Observe that the image of each fibre of $\mathcal{L}$ in $\mathcal{V}$ is a square lattice. This shows that the family is isotrivial. To see that it is not a trivial family, observe that if it were then $\mathcal{L}$ would have to be a trivial Λ-fibred bundle. But if it were, the transition function $t_{U,V}$ would extend continuously to all of $\mathbb{Z}^2 \times \mathbb{C}^\times$. Clearly it does not. This proves that there is no moduli space for lattices.

Observe that the obstruction to the existence of a moduli space was the automorphism

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{\varphi} & \mathbb{C} \\ \| & & \downarrow{\scriptstyle f} \\ \mathbb{Z}^2 & \xrightarrow{\varphi} & \mathbb{C}, \end{array}$$

where f sends z to $-z$, of the lattice given by $\varphi : (m, n) \mapsto m + in$. If only a few lattices had nontrivial automorphisms, we might approach the moduli problem by ignoring these and constructing a moduli space for lattices with only the identity automorphism. All lattices, however, have this negation automorphism. This is a significant barrier to solving our moduli problem.

There are a number of other standard approaches to dealing with obstructions to the existence of a moduli space. One is to relax the requirement that there exist a universal family, and look instead for a space whose points are in bijection with isomorphism classes of objects in some sensible way. We call such a space a coarse moduli space, and in due time we will construct one for lattices. Although this classifies objects, it is not so useful for classifying families. For example, a single point is a coarse moduli space for vector bundles of a given dimension, but this does not begin to capture the variety of families—that is, vector bundles—that exist. Nonetheless, we will consider this approach in Chapter 3.

Another approach is to rigidify our objects. This will also be introduced in Chapter 3, but will come to the fore as the focus of Chapter 4. Here we introduce extra structure to remove automorphisms. An advantage of this approach is that we can then look at ways to minimise this structure, and construct some close approximations to a moduli space, even if we don't quite make it to a moduli space itself.

Before we do any of this, however, we demonstrate that the moduli problem for lattices is in a very precise sense equivalent to the moduli problem for elliptic curves, and hence worth discussing further.

Chapter 2

Equivalent Moduli Problems

Varying a lattice can simply be thought of as holomorphically varying a pair of generators for the lattice as points in the complex plane. It is geometrically less clear what it means to vary a complex structure on an elliptic curve. Due to this, one might expect that the moduli problem for lattices is more intuitive, and hence easier to work with, than that for elliptic curves. The moduli problems are in fact equivalent: any moduli space for lattices is also a moduli space for elliptic curves. In this chapter we define precisely what it means for moduli problems to be equivalent, and outline an argument that this is indeed so for elliptic curves and lattices. This will allow us, in later chapters, to work with lattices to come to an understanding of families of elliptic curves.

2.1 Equivalent Moduli Problems

Suppose we are given two moduli problems. Further suppose that the categories of base spaces are the same, and that there exists a natural isomorphism between the moduli functors. Then, as the composition of natural isomorphisms is again a natural isomorphism, a moduli space for one functor is the same as a moduli space for the other. In this case we call the two moduli problems *equivalent*.

We unpack what this means a little. Let

$$S, T : \mathbf{Spaces} \longrightarrow \mathbf{Set}$$

be moduli functors, and $\tau : S \dot{\to} T$ be a natural isomorphism. The natural isomorphism τ assigns a set bijection $\tau_B : S(B) \to T(B)$ to each base space B in such a way that for any morphism $f : B_1 \to B_2$ of base spaces the naturality square

$$\begin{array}{ccc} S(B_2) & \xrightarrow{\tau_{B_2}} & T(B_2) \\ {\scriptstyle S(f)}\downarrow & & \downarrow{\scriptstyle T(f)} \\ S(B_1) & \xrightarrow{\tau_{B_1}} & T(B_1) \end{array}$$

commutes. Recall that the sets $S(B_i)$ and $T(B_i)$ consist of isomorphism classes of families over B_i, and that the functions $S(f)$ and $T(f)$ are pullbacks of families along the map $f : B_1 \to B_2$ of base spaces. Thus two moduli problems are equivalent if and only if, for any given base space, there is a bijection between the two types of families over that base space, and these bijections commute with pullbacks.

In this chapter we shall construct such bijections between isomorphism classes of families of lattices and isomorphism classes of families of elliptic curves.

2.2 Elliptic Curves and Lattices

We wish to construct an explicit natural isomorphism between the functors $\mathscr{E}\mathit{ll}$ and $\mathscr{L}\mathit{att}$. To do this we first understand the relationship between elliptic curves and lattices on an object by object level.

Periods

Given a closed 1-form on a compact Riemann surface X, integration of this form along elements of the first homology group defines an element of the first cohomology group with coefficients in $\mathbb{C}$. It is from this fact that we will construct a map from the set of elliptic curves to that of lattices. We first review the basic theory. The results and arguments here are mainly taken from [11, Ch.III §3].

Proposition 2.1. *Let ω be a closed differential 1-form on a Riemann surface X. Then the map*

$$\begin{aligned} \textstyle\int \omega : H_1(X;\mathbb{Z}) &\longrightarrow \mathbb{C}; \\ [\gamma] &\longmapsto \int_\gamma \omega, \end{aligned}$$

where $[\gamma]$ is the homology class of a closed curve γ on X, is well-defined.

Proof. Let γ, γ' be paths in $[\gamma]$. Then the concatenation of paths $\gamma - \gamma'$ is a contractible loop in X, and so can be expressed as the boundary $\partial\Omega$ of a simply connected region Ω in X. Stokes' theorem then gives

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\gamma-\gamma'} \omega = \int_{\partial\Omega} \omega = \iint_\Omega d\omega = 0.$$

This proves that the value of $\int \omega$ at $[\gamma]$ is independent of choice of representative γ. □

We call the elements of the image of this map the *periods* of ω. Suppose ω is an exact 1-form, so there exists a smooth function f such that $df = \omega$. Then, observing that for any loop $\gamma : [0,1] \to X$ in X we have $\gamma(0) = \gamma(1)$, we see

$$\int_\gamma \omega = \int_\gamma df = f(\gamma(1)) - f(\gamma(0)) = 0.$$

Thus an exact form only has the trivial period. More relevant to us is the fact that the converse is also true.

Proposition 2.2. *Let ω be a closed differential 1-form on a Riemann surface X. If the map $\int \omega$ is the zero map, then ω is exact.*

Proof. We shall construct a smooth map f such that $df = \omega$ by integrating. Fix some basepoint $p_0 \in X$, and observe that any element of $H_1(X;\mathbb{Z})$ may be represented as the homotopy class of a loop based at p_0. Define $f : X \to \mathbb{C}$ by

$$f(p) = \int_{p_0}^{p} \omega.$$

This value is unique up to $\int_\gamma \omega$ for elements $[\gamma] \in H_1(X;\mathbb{Z})$. But for all such γ, $\int_\gamma \omega = 0$ by hypothesis. Thus f is well-defined.

The fundamental theorem of calculus then implies that $df = \omega$, which in turn implies that f is smooth. The fact that $df = \omega$ then also implies that ω is exact, as claimed. □

Recall that we write $H^0(X, \Omega^1_X)$ for the vector space of holomorphic 1-forms on X. We write $\overline{H^0(X, \Omega^1_X)}$ for the vector space of complex conjugates of holomorphic 1-forms—these are the so-called antiholomorphic 1-forms.

Lemma 2.3. *Let X be a compact Riemann surface. Then the map*

$$\begin{aligned} H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)} &\longrightarrow H^1(X;\mathbb{C}); \\ (\omega_1, \overline{\omega}_2) &\longmapsto \textstyle\int \omega_1 + \overline{\omega}_2 \end{aligned}$$

is injective.

Proof. We prove that the kernel of the map is trivial. Suppose we have holomorphic 1-forms ω_1, ω_2 such that $\int \omega_1 + \overline{\omega}_2$ is the zero map. We wish to show that $\omega_1 = \omega_2 = 0$. We prove that $\omega_1 = 0$. The fact that $\omega_2 = 0$ follows similarly.

Locally we may write $\omega_1 = g_1 dz$, where g_1 is holomorphic. Observe that in this chart

$$i\omega_1 \wedge \overline{\omega}_1 = ig_1(dx + idy) \wedge \overline{g_1}(dx - idy) = 2|g_1|^2 dx \wedge dy,$$

where $dz = dx + idy$ is the local coordinate. This is a nonnegative function times the volume element, so

$$i \iint_X \omega_1 \wedge \overline{\omega}_1 = 0$$

if and only if ω_1 is exactly zero.

By Proposition 2.2, there exists a smooth function f such that $df = \omega_1 + \overline{\omega}_2$. Again working locally, $\omega_1 = g_1 dz$ and $\omega_2 = g_2 dz$ for holomorphic g_1, g_2, and so

$$\omega_1 \wedge \omega_2 = g_1 dz \wedge g_2 dz = 0.$$

We thus see that

$$\omega_1 \wedge \overline{\omega}_1 = \omega_1 \wedge \overline{\omega}_1 + \omega_1 \wedge \omega_2 = \omega_1 \wedge (\overline{\omega_1 + \overline{\omega}_2})) = \omega_1 \wedge d\overline{g}.$$

Also observe that since ω_1 is holomorphic, it is closed, and so

$$d(\overline{g}\omega_1) = d\overline{g} \wedge \omega_1 + \overline{g} d\omega_1 = d\overline{g} \wedge \omega_1.$$

Thus

$$i \iint_X \omega_1 \wedge \overline{\omega}_1 = i \iint_X \omega_1 \wedge d\overline{g} = i \iint_X d(\overline{g}\omega_1) = 0,$$

where the last equality holds using Stokes' theorem and the compactness of the surface X. This proves that $\omega_1 = 0$, which is enough to prove the proposition. □

The above is all we shall require for what follows. For fun, however, and because they are within close reach, we close this section by proving two fundamental theorems in the case of compact Riemann surfaces.

Corollary 2.4 (Hodge decomposition)**.** *Let X be a compact Riemann surface. Then*

$$H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)} \cong H^1(X; \mathbb{C}).$$

Proof. Lemma 2.3 shows that we have an injective map between these two groups. By inspection, this map is in fact a map of complex vector spaces. Now Proposition 0.17 shows the dimension of the left-hand side is $2g(X)$, while standard results in the computation of homology groups and Poincaré duality show the dimension of the right-hand side is also $2g(X)$. Thus the map is an isomorphism of complex vector spaces, and hence, a fortiori, one of groups. □

We define the ith De Rham cohomology group $H^i_{DR}(X)$ of a smooth manifold X to be the group of closed differential i-forms modulo the exact differential i-forms. Propositions 2.1, 2.2 and the discussion between the two show we have an injective map

$$H^1_{DR}(X) \longrightarrow H^1(X;\mathbb{C}).$$

On the other hand, the isomorphism

$$H^0(X,\Omega^1_X) \oplus \overline{H^0(X,\Omega^1_X)} \longrightarrow H^1(X;\mathbb{C})$$

factors through this map, sending $(\omega_1, \overline{\omega}_2)$ to $[\omega_1 + \overline{\omega}_2]$ to $\int \omega_1 + \overline{\omega}_2$. Thus the previous map is an isomorphism. This proves De Rham's theorem for compact Riemann surfaces.

Corollary 2.5 (De Rham's theorem)**.** *Let X be a compact Riemann surface. Then*

$$H^1_{DR}(X) \cong H^1(X;\mathbb{C}).$$

Both these theorems hold more generally and are of great significance. See Carlson, Müller-Stach and Peters [5, Ch.2] and Bott and Tu [2, Ch.II] respectively for more details.

Elliptic Curves and Lattices

It follows from the uniformisation theorem for Riemann surfaces that up to isomorphism all compact Riemann surfaces of genus 1 may be written as a quotient $V/\varphi(\Lambda)$, where (V,φ) is a lattice. An exposition of this can be found in Farkas and Kra [8, §§IV.4-IV.5], or Silverman [29, §1.4]. Here we provide an alternate proof of this fact, explicitly constructing a biholomorphism using the map defined in the above section. We favour this approach as this allows us to give explicit interpretations of our additional structures on lattices in terms of structures on elliptic curves. This section owes much to Hain [14, §1]; in particular the proofs of Propositions 2.9 and 2.10 are based on arguments found there.

Given a lattice, it is not difficult to construct an elliptic curve. Note first that any 1-dimensional complex vector space V has a canonical complex structure: any vector $v \in V$ induces a complex vector space isomorphism $V \cong \mathbb{C}$ by sending v to 1, and this isomorphism gives a complex coordinate neighbourhood for all of V. For any two choices of v, the coordinate neighbourhoods induced by v are compatible; in this way this structure is canonical.

Theorem 2.6. *Let (V, φ) be a lattice. Then $(V/\varphi(\Lambda); 0)$ is an elliptic curve.*

Proof. The discrete subgroup $\varphi(\Lambda)$ of V acts freely and properly discontinuously on the 1-dimensional complex manifold V by vector addition, so Lemma 0.15 gives a natural 1-dimensional complex structure on $V/\varphi(\Lambda)$. This quotient is toplogically a torus, and so has genus 1. Thus $(V/\varphi(\Lambda); 0)$ is an elliptic curve. □

Conversely, from each elliptic curve we can construct a unique lattice. Note that in the above construction the vector space V becomes the universal cover of the elliptic curve $V/\varphi(\Lambda)$. Since $V/\varphi(\Lambda)$ can be viewed as a compact Riemannian manifold by choosing some isomorphism of V with $\mathbb{C}$, it is geodesically complete, and hence the exponential map canonically identifies the universal cover V with the tangent space at the marked point $T_0(V/\varphi(\Lambda))$. Furthermore, the action of Λ on V given by the map φ and the additive structure on V identifies Λ with the fundamental group $\pi_1(V/\varphi(\Lambda); 0)$. Since Λ is abelian, this then induces an isomorphism of Λ with the first homology group $H_1(V/\varphi(\Lambda); \mathbb{Z})$. Thus the lattice map $\varphi : \Lambda \to V$ naturally corresponds to a map

$$H_1(V/\varphi(\Lambda); \mathbb{Z}) \longrightarrow T_0(V/\varphi(\Lambda))$$

from the first homology group to the tangent space at the marked point. Motivated by this, given an elliptic curve $(E; O)$, we shall construct a lattice given by a map $H_1(E; \mathbb{Z}) \to T_O E$ and such that the elliptic curve constructed from this lattice is isomorphic to the given elliptic curve.

This lattice is the so-called period lattice of the elliptic curve. In the previous subsection we showed that there exists a pairing

$$H_1(E; \mathbb{Z}) \otimes H^0(E, \Omega^1_E) \xrightarrow{\int} \mathbb{C},$$

and hence a $\mathbb{Z}$-linear map

$$H_1(E; \mathbb{Z}) \longrightarrow (H^0(E, \Omega^1_E))^*,$$

where $(H^0(X,\Omega_X^1))^*$ is the dual of the space of holomorphic 1-forms on X. I claim that $(H^0(E,\Omega_E^1))^*$ may naturally be identified with the tangent space $T_O E$. We show this after we show the following lemma.

Lemma 2.7. *Let ω be a nonzero holomorphic 1-form on an elliptic curve. Then ω is nowhere zero.*

Proof. By the Poincaré-Hopf formula (Proposition 0.18), the degree of ω is $0 = 2g-2$, where g is the genus of the elliptic curve. Since ω is holomorphic, it has no poles, and thus its degree is the sum of the orders of its zeroes. But these are each positive integers. Thus ω can have no zeroes. □

Proposition 2.8. *Let $(E;O)$ be an elliptic curve. Then the dual $(H^0(E,\Omega_E^1))^*$ of the space of holomorphic 1-forms is canonically isomorphic to the tangent space $T_O E$ at the marked point of E.*

Proof. We shall show that the space $H^0(E,\Omega_E^1)$ of holomorphic 1-forms on E is canonically isomorphic to the cotangent space $T_O^* E$ at the marked point of E. By taking duals this is equivalent to the lemma.

The key fact is that the global holomorphic 1-forms on E are naturally identified with the sections of the holomorphic cotangent bundle T^*E of E. This is true because the holomorphic cotangent bundle is the bundle associated to the cocycle $(dz_j/dz_i)_{i,j\in I}$, where $(U_i,z_i)_{i\in I}$ is a coordinate covering of E, and this cocycle describes precisely the transitions of 1-forms between charts. Let now ω be a nonzero holomorphic 1-form on E. Then by Lemma 2.7, ω is nowhere zero. Thus ω gives a nowhere zero section of $s_\omega : E \to T^*E$ of T^*E. Evaluating s_ω at O, we thus have a nonzero element of T^*E. This process defines a nonzero $\mathbb{C}$-linear map

$$H^0(E,\Omega_E^1) \longrightarrow T_O^* E.$$

Since both the domain and codomain of this map are 1-dimensional complex vector spaces, this map must hence be an isomorphism. This proves the proposition. □

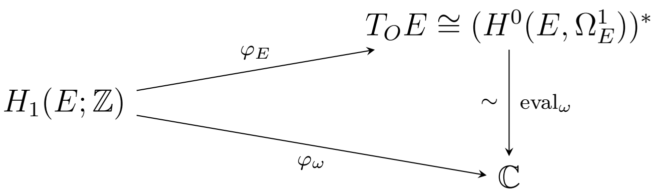
We thus have a well-defined $\mathbb{Z}$-linear map

$$\varphi_E : H_1(E;\mathbb{Z}) \longrightarrow T_O E.$$

We now wish to show that $(T_O E, \varphi_E)$ is a lattice with the properties we have claimed. For this we will find it easiest to work with respect to a basis. Given any nonzero $\omega \in H^0(E, \Omega^1_E)$, the map

$$\begin{aligned} \mathrm{eval}_\omega : \quad & (H^0(E, \Omega^1_E))^* \longrightarrow \mathbb{C}; \\ & \Big[f : H^0(E, \Omega^1_E) \to \mathbb{C}\Big] \longmapsto f(\omega) \end{aligned}$$

is an isomorphism of vector spaces. Write φ_ω for the map $\int \omega : H_1(E; \mathbb{Z}) \to \mathbb{C}$ defined in the previous section. Then



commutes. In particular, this shows that $(T_O E, \varphi_E)$ is a lattice if and only if $(\mathbb{C}, \varphi_\omega)$ is, and that $(T_O E/\varphi_E(H_1(E; \mathbb{Z})); 0)$ is isomorphic to $(E; O)$ if and only if $(\mathbb{C}/\varphi_\omega(H_1(E; \mathbb{Z})); 0)$ is.

Proposition 2.9. *Given an elliptic curve $(E; O)$, and any nonzero holomorphic 1-form ω on E, the pair $(\mathbb{C}, \varphi_\omega)$ is a lattice.*

Proof. To begin, observe that since E is of genus 1, its first homology group $H_1(E; \mathbb{Z})$ is a rank two free abelian group. Also observe that $\mathbb{C}$ is a 1-dimensional complex vector space. Thus the domain and codomain of the map $\varphi_\omega : H_1(E; \mathbb{Z}) \to \mathbb{C}$ are of the required form.

It now suffices to show that, given generators γ_1, γ_2 of $H_1(E; \mathbb{Z})$, their images

$$\lambda_1 := \int_{\gamma_1} \omega, \qquad \lambda_2 := \int_{\gamma_2} \omega$$

are $\mathbb{R}$-linearly independent. For the sake of contradiction, suppose to the contrary that there is a real number $k \in \mathbb{R}$ such that $\lambda_2 = k\lambda_1$.

Tensoring with $\mathbb{R}$, we may extend φ_ω to a map $H_1(E; \mathbb{R}) \to \mathbb{C}$. Under this map

$$\int_{k\gamma_1 - \gamma_2} \omega = k \int_{\gamma_1} \omega - \int_{\gamma_2} \omega = k\lambda_1 - \lambda_2 = 0.$$

Since k is a real scalar, $\overline{k} = k$, and thus we also have

$$\int_{k\gamma_1 - \gamma_2} \overline{\omega} = k \int_{\gamma_1} \overline{\omega} - \int_{\gamma_2} \overline{\omega} = \overline{k \int_{\gamma_1} \omega - \int_{\gamma_2} \omega} = \overline{k\lambda_1 - \lambda_2} = 0.$$

This implies the linear subspace $L = \{a(k\gamma_1 - \gamma_2) \mid a \in \mathbb{R}\}$ of $H_1(E;\mathbb{R})$ lies in the kernel of both the maps $\int \omega$ and $\int \overline{\omega}$, and so we may consider them both as linear maps $H_1(E;\mathbb{R})/L \to \mathbb{C}$. Since these are maps from a 1-dimensional real vector space into $\mathbb{C}$, they must be $\mathbb{C}$-linearly dependent. This implies that there exists $\alpha \in \mathbb{C}$ such that the map $\int \omega + \alpha\overline{\omega} : H_1(E;\mathbb{Z}) \to \mathbb{C}$ is the zero map.

But in Lemma 2.3 we saw that the map

$$\begin{aligned} H^0(E, \Omega^1_E) \oplus \overline{H^0(E, \Omega^1_E)} &\longrightarrow H^1(E;\mathbb{C}); \\ (\omega_1, \overline{\omega}_2) &\longmapsto \int \omega_1 + \overline{\omega}_2 \end{aligned}$$

is injective. This shows that ω is zero, a contradiction. Hence the pair $(\mathbb{C}, \varphi_\omega)$ is a lattice. □

We have thus given a construction that associates to each elliptic curve a unique lattice. We know that from this lattice, we can construct an elliptic curve. This elliptic curve is, as we have desired, isomorphic to the one we started with.

Proposition 2.10. *Let $(E; O)$ be an elliptic curve. Then for any choice of nonzero holomorphic 1-form ω on E, the elliptic curve $(\mathbb{C}/\varphi_\omega(H_1(E;\mathbb{Z})); 0)$ is isomorphic to $(E; O)$.*

Proof. Define the map

$$\begin{aligned} f : E &\longrightarrow \mathbb{C}/\varphi_\omega(H_1(E;\mathbb{Z})); \\ z &\longmapsto \int_O^z \omega \mod \varphi_\omega(H_1(E;\mathbb{Z})). \end{aligned}$$

Observe that for any path from O to x the integral $\int_O^x \omega$ is well-defined up a choice of homotopy class for the path, and distinct homotopy classes differ only by elements of $H_1(E;\mathbb{Z})$. Thus $\int_O^x \omega$ takes a unique value modulo $\varphi_\omega(H_1(E;\mathbb{Z}))$, and so f is well-defined.

Since f is defined as the integral of ω, $df = \omega$, and so f is a holomorphism. In fact, as ω is nowhere zero (Proposition 2.7), f is a local biholomorphism. On the other hand, applying the identity theorem to f we see that every fibre of f is a discrete set, and since it is a map of compact Riemann surfaces this implies that every fibre of f is finite. It follows from these two facts that f is a covering map.

Observe, however, the induced map $f_* : H_1(X;\mathbb{Z}) \to H_1(\mathbb{C}/\varphi_\omega(H_1(E;\mathbb{Z}));\mathbb{Z})$ on homology groups is an isomorphism by the construction of φ_ω. Since these are each isomorphic to the respective fundamental groups of the elliptic curves, covering space theory implies that X and $\mathbb{C}/\varphi_\omega(H_1(E;\mathbb{Z}))$ are isomorphic as Riemann surfaces. Since f carries O to 0, this shows $(E;O)$ and $(\mathbb{C}/\varphi_\omega(H_1(E;\mathbb{Z})),0)$ are isomorphic, as claimed. □

Observe that it follows from this proposition and Lemma 0.15 that the universal cover of every elliptic curve is $\mathbb{C}$. This is the uniformisation theorem for elliptic curves.

We have now shown the following theorem.

Theorem 2.11. *Let $(E;O)$ be an elliptic curve. Then $(T_O E, \varphi_E)$ is a lattice, and $(T_O E/\varphi_E(H_1(E;\mathbb{Z}));0)$ is an elliptic curve canonically isomorphic to $(E;O)$.*

In particular, this gives a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of elliptic curves} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of lattices} \end{array} \right\}.$$

As Mazur outlines in [23, §5], here we have demonstrated the basics of Hodge theory. It can be shown that for any compact Riemann surface X of genus g the bilinear map

$$(\omega_1, \omega_2) \longmapsto i \iint_X \omega_1 \wedge \overline{\omega_2}$$

defines a Hermitian form on the g-dimensional complex vector space $H^0(X, \Omega^1_X)$. Furthermore, it can also be shown that functionals

$$\omega \longmapsto \int_\gamma \omega$$

formed by the integration of 1-forms with respect to elements of the first homology group define an injective map

$$H_1(X;\mathbb{Z}) \longrightarrow (H^0(X, \Omega^1_X))^*$$

from the homology group to the dual space of $H^0(X, \Omega^1_X)$. This map is then an injective group homomorphism from a free abelian group of rank $2g$ to a complex vector space of dimension g whose image spans $(H^0(X, \Omega^1_X))^*$ as a real vector space. We shall say it defines a $2g$-dimensional lattice.

This data—the Hermitian form and the $2g$-dimensional lattice—in fact allow us to recover the Riemann surface X up to isomorphism. Note that in the genus 1 case there is only one such Hermitian form, and so we may disregard it. As we have seen, in the genus 1 case it is also true that for every lattice there exists a Riemann surface that produces it. This is not true in general.

Aside: The Marked Point

Observe that the marked point comes from the identity of the additive group V. Since the group identity is preserved by morphisms of lattices, the analogy between lattices and elliptic curves requires that the marked point be preserved by morphisms of elliptic curves. It is hence important to keep track of this point when constructing an elliptic curve.

In particular, our definitions of elliptic curve and lattice are constructed to correspond precisely. One might wonder why, however, we did not choose to define our lattices differently, say as an action of Λ on a 1-dimensional complex affine space. This would remove the need for a marked point. The reason for this is to reduce automorphisms.

Proposition 2.12. *Let E be a compact Riemann surface of genus 1, and let $O, P \in E$. Then the elliptic curves $(E; O)$ and $(E; P)$ are isomorphic.*

Proof. Since E is isomorphic to a quotient of $\mathbb{C}$ by a lattice, it has a natural complex Lie group structure induced by the group structure on $\mathbb{C}$. Let $+_O$ be the isomorphism mapping a point $x \in E$ to the point $x + O \in E$, and similarly for P. Then $+_P \circ +_O^{-1}$ is a holomorphic automorphism of E mapping O to P. This proves that $(E; O)$ and $(E; P)$ are isomorphic. □

We have already noted that automorphisms are undesirable as they complicate our moduli problem. The above proposition says that each compact Riemann surface of genus 1 has an uncountable set of automorphisms indexed by its own points. By marking a point, we remove all of these automorphisms. Although we still have some automorphisms, we will see in the next chapter that we now only have finitely many. Note too that this proposition also shows that which point we choose to mark is not particularly important: we care only that we have a marked point.

Maps of Elliptic Curves and Maps of Lattices

Since we now have a very concrete general form for elliptic curves—every elliptic curve is isomorphic to one of the form $(\mathbb{C}/\varphi(\Lambda); 0)$ for some lattice $(\mathbb{C}, \varphi)$—we can concretely compute the maps between them. These maps are already familiar to us.

Proposition 2.13. *Let $(\mathbb{C}/\varphi_1(\Lambda); 0)$, $(\mathbb{C}/\varphi_2(\Lambda); 0)$ be elliptic curves. Then every* *map* $f : (\mathbb{C}/\varphi_1(\Lambda); 0) \to (\mathbb{C}/\varphi_2(\Lambda); 0)$ *of elliptic curves is of the form*

$$\begin{aligned} f : \mathbb{C}/\varphi_1(\Lambda) &\longrightarrow \mathbb{C}/\varphi_2(\Lambda); \\ z + \varphi_1(\Lambda) &\longmapsto \alpha z + \varphi_2(\Lambda) \end{aligned}$$

for some $\alpha \in \mathbb{C}$.

Proof. Note that f maps $0 + \varphi_1(\Lambda)$ to $0 + \varphi_2(\Lambda)$. Standard lifting results in covering space theory allow us to lift f to a holomorphic function $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ such that $\widetilde{f}(0) = 0$ and

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widetilde{f}} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\varphi_1(\Lambda) & \xrightarrow{f} & \mathbb{C}/\varphi_2(\Lambda) \end{array}$$

commutes, where the vertical maps are the quotient maps. It thus suffices to find $\alpha \in \mathbb{C}$ such that $\widetilde{f}$ is of the form $z \mapsto \alpha z$. In order to do this we shall show that the derivative $\widetilde{f}'$ of $\widetilde{f}$ is constant with value α.

Observe that for all $\lambda \in \varphi_1(\Lambda)$ and all $z \in \mathbb{C}$, we have

$$\widetilde{f}(z) - \widetilde{f}(z + \lambda) \in \varphi_2(\Lambda).$$

But this difference is a continuous function of z and $\varphi_2(\Lambda)$ is a discrete set, so it must be constant. Differentiating, we hence find that, for all $\lambda \in \varphi_1(\Lambda)$,

$$\widetilde{f}'(z) = \widetilde{f}'(z + \lambda).$$

Thus $\widetilde{f}'$ is a doubly periodic function with period lattice $\varphi_1(\Lambda)$. Due to this, we may consider $\widetilde{f}'$ as a function $\mathbb{C}/\varphi_1(\Lambda) \to \mathbb{C}$. Then since $\mathbb{C}/\varphi_1(\Lambda)$ is compact, $\widetilde{f}'$ is bounded, and hence by Liouville's theorem is of constant value $\alpha \in \mathbb{C}$.

Thus $\widetilde{f}$ is of the form $z \mapsto \alpha z + \beta$, where $\alpha, \beta \in \mathbb{C}$. But $\widetilde{f}(0) = 0$, so we may conclude that $\beta = 0$, and so f is of the form specified. □

For any $\alpha \in \mathbb{C}$ such that $\alpha\varphi_1(\Lambda) \subseteq \varphi_2(\Lambda)$, it is clear that the map $z \mapsto \alpha z$ descends to a map $(\mathbb{C}/\varphi_1(\Lambda); 0) \to (\mathbb{C}/\varphi_2(\Lambda); 0)$. Thus we have the bijection

$$\{\alpha \in \mathbb{C} \mid \alpha\varphi_1(\Lambda) \subseteq \varphi_2(\Lambda)\} \longleftrightarrow \left\{ \begin{array}{c} \text{maps of elliptic curves} \\ (\mathbb{C}/\varphi_1(\Lambda); 0) \to (\mathbb{C}/\varphi_2(\Lambda); 0) \end{array} \right\}.$$

Recall, on the other hand, that we already have

$$\{\alpha \in \mathbb{C} \mid \alpha\varphi_1(\Lambda) \subseteq \varphi_2(\Lambda)\} \longleftrightarrow \left\{ \begin{array}{c} \text{maps of lattices} \\ (\mathbb{C}, \varphi_1) \to (\mathbb{C}, \varphi_2) \end{array} \right\}.$$

Thus we have a bijection

$$\left\{ \begin{array}{c} \text{maps of elliptic curves} \\ (\mathbb{C}/\varphi_1(\Lambda); 0) \to (\mathbb{C}/\varphi_2(\Lambda); 0) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{maps of lattices} \\ (\mathbb{C}, \varphi_1) \to (\mathbb{C}, \varphi_2) \end{array} \right\}.$$

In categorical terms, we may sum up this section by saying that we have constructed a functor from the category of lattices to the category of elliptic curves. The fact that every elliptic curve is isomorphic to the image of some lattice under this map then implies this functor is *essentially surjective*, and the fact that there is a bijection between the set of maps of lattices and the set of maps of their corresponding elliptic curves implies that this functor is *fully faithful*. By a basic result in category theory (see [22, Ch. IV §4]), this shows that the categories of lattices and of elliptic curves are equivalent. Given our claims that the moduli problems for these two objects are the same, this should be entirely unsurprising.

2.3 A Moduli Problem Equivalence

We have now seen that lattices and elliptic curves are essentially the same thing on an object by object basis. What we really want to show, however, is that our ideas for their deformations agree: that the moduli problems are equivalent. By the category equivalence proved in the previous section, we may interpret this as the statement that a holomorphic deformation of an elliptic curve is precisely a holomorphic deformation its period lattice.

Recall that to show the moduli problems are equivalent it suffices to show that there exists a bijection between isomorphism classes of families of lattices and isomorphism classes of families of elliptic curves that commutes with pullbacks. In this section we indicate how to construct such a bijection and outline an argument that it does indeed have these properties.

From Families of Lattices to Families of Elliptic Curves

Let $(\mathcal{V}, \mathcal{L}, B, \Phi)$ be a family of lattices over a complex manifold B. Over each $x \in B$, the fibre $(\mathcal{V}_x, \mathcal{L}_x, \Phi|_{\mathcal{L}_x})$ of this family is a lattice. We are thus able to construct an elliptic curve by taking the quotient of the vector space $\mathcal{V}_x$ by the discrete subgroup $\Phi(\mathcal{L}_x)$. Repeating this construction for each point x, we associate to each point in B an elliptic curve. We shall show that this collection of elliptic curves naturally forms a family of elliptic curves.

In order to do this, we firstly give a precise definition of the quotient manifold. Define an equivalence relation on the vector bundle $\mathcal{V}$ by calling two points u, v of $\mathcal{V}$ equivalent if and only if $\pi_{\mathcal{V}}(u) = \pi_{\mathcal{V}}(v)$ and $u - v$ lies in $\Phi(\mathcal{L}_{\pi(u)})$—that is, if and only if they lie in the same fibre and they differ by an element of the lattice. We denote the quotient space $\mathcal{V}/\Phi(\mathcal{L})$. Observe that $\mathcal{V}/\Phi(\mathcal{L})$ comes equipped with a map $\pi : \mathcal{V}/\Phi(\mathcal{L}) \to B$ sending the equivalence class of a point $v \in \mathcal{V}$ to $\pi_{\mathcal{V}}(v)$. This is well-defined as for $u, v \in \mathcal{V}$ to lie in the same equivalence class we must have $\pi_{\mathcal{V}}(u) = \pi_{\mathcal{V}}(v)$.

Lemma 2.14. *Let $(\mathcal{V}, \mathcal{L}, B, \Phi)$ be a family of lattices over a complex manifold B. The quotient space $\mathcal{V}/\Phi(\mathcal{L})$ has a natural complex structure such that the quotient map $\mathcal{V} \to \mathcal{V}/\Phi(\mathcal{L})$ is holomorphic.*

Proof. This is a local issue. Since a trivialising neighbourhood for the family of lattices exists around any point of the base space B, it suffices to show that the lemma is true for $\mathcal{V} = V \times B$ and $\mathcal{L} = \Lambda \times B$, where V is a 1-dimensional complex vector space and Λ is a rank two free abelian group.

Observe that the equivalence relation on $\mathcal{V}$ may be described on each fibre $\mathcal{V}_x$ by the action of the group $\mathcal{L}_x$. But in this case each group $\mathcal{L}_x$ may be canonically identified with Λ. Thus we may describe $\mathcal{V}/\Phi(\mathcal{L})$ as the quotient of $\mathcal{V}$ by an action of the group Λ. Since each $\mathcal{L}_x$ acts freely and properly discontinuously on the fibre $\mathcal{V}_x$, and the base space B is Hausdorff, Λ acts freely and properly discontinuously on $\mathcal{V}$. The result then follows from Lemma 0.15. □

It is clear that π is surjective. Since $\mathcal{V}$ is a covering of $\mathcal{V}/\Phi(\mathcal{L})$, π locally behaves like $\pi_{\mathcal{V}}$ and hence, since the projection of a line bundle $\pi_{\mathcal{V}}$ is a submersion, π is also a submersion. Observe also that the zero section $B \to \mathcal{V}$ of the vector bundle $\mathcal{V}$—which is holomorphic since $\mathcal{V}$ is a holomorphic vector bundle—induces a holomorphic section $s : B \to \mathcal{V}/\Phi(\mathcal{L})$ of π by composition with the holomorphic map $\mathcal{V} \to \mathcal{V}/\Phi(\mathcal{L})$. Since $\pi^{-1}(x) = \mathcal{V}_x/\Phi(\mathcal{L}_x)$, the fibre $(\pi^{-1}(x), s(x))$ is an elliptic curve for each $x \in B$. We have verified that the data $(\mathcal{V}/\Phi(\mathcal{L}), B, \pi, s)$ specifies a family of elliptic curves over B. To summarise:

Proposition 2.15. *Let $(\mathcal{V}, \mathcal{L}, B, \Phi)$ be a family of lattices over a complex manifold B. Then $(\mathcal{V}/\Phi(\mathcal{L}), B, \pi, s)$ is a family of elliptic curves over B.*

It in straightforward to check that morphisms of families of lattices descend to morphisms of families of elliptic curves under this quotient construction, and hence isomorphic families of lattices give rise to isomorphic families of elliptic curves. We thus have a map from the set of isomorphism classes of families of lattices to that of elliptic curves. As the construction given is a relative quotient of group bundles, it is also not difficult to check that this map commutes with pullbacks.

From Families of Elliptic Curves to Families of Lattices

It remains to show that the above map is a bijection. For this we construct an inverse map. Let $(\mathcal{E}, B, \pi, s)$ be a family of elliptic curves over a complex manifold B. Using the fact that the fibre over each point of B is an elliptic curve, we shall build a Λ-fibred bundle from the homology groups of each fibre, and a line bundle from the tangent spaces at the marked point of each fibre. The period map of Theorem 2.11 then gives a map between these two bundles, and it can be checked that this data gives a family of lattices over B.

We first construct the Λ-fibred bundle. Define

$$\mathcal{H}_1(\mathcal{E};\mathbb{Z}) := \bigcup_{x \in B} H_1(\mathcal{E}_x;\mathbb{Z}).$$

There exists a canonical projection $\mathcal{H}_1(\mathcal{E};\mathbb{Z}) \to B$ mapping each element of $H_1(\mathcal{E}_x;\mathbb{Z})$ to x. We wish to give this set a topology and a complex structure in such a way that it forms a Λ-fibred bundle.

We have seen that every family of elliptic curves is locally trivial as a differentiable fibre bundle. Let $\{U_\alpha\}$ be a local trivialisation of $\mathcal{E}$ consisting of contractible open sets. Then for any U_α and any $x \in U_\alpha$ we have

$$\mathcal{E}_{U_\alpha} \cong \mathcal{E}_x \times U_\alpha$$

as smooth manifolds. Since U_α is contractible, the inclusion $\mathcal{E}_x \hookrightarrow \mathcal{E}_{U_\alpha}$ is a homotopy equivalence, and so this induces a natural isomorphism

$$H_1(\mathcal{E}_x;\mathbb{Z}) \cong H_1(\mathcal{E}_{U_\alpha};\mathbb{Z}).$$

In particular, fixing a point $x_0 \in U_\alpha$, this gives a natural isomorphism between the homology group $H_1(\mathcal{E}_{x_0};\mathbb{Z})$ of the fibre $\mathcal{E}_{x_0}$ and the homology group $H_1(\mathcal{E}_x;\mathbb{Z})$ of the fibre $\mathcal{E}_x$ over any point $x \in U_\alpha$. This allows us to put a complex structure on $\mathcal{H}_1(\mathcal{E};\mathbb{Z})$ such that for every open set U_α in the above cover and every $x_0 \in U_\alpha$ we have

$$\bigcup_{x \in U_\alpha} H_1(\mathcal{E}_x;\mathbb{Z}) \cong H_1(\mathcal{E}_{x_0};\mathbb{Z}) \times U_\alpha.$$

With the natural group structure on each homology group $H_1(\mathcal{E}_x;\mathbb{Z})$ this forms a group bundle over B. We call this bundle the *relative first homology bundle*.

Observing that there is a natural injection $TB \hookrightarrow s^*TE$, we define the *normal bundle of the section* $s : B \to \mathcal{E}$ by the quotient

$$N_{\mathcal{E}}B := s^*T\mathcal{E}/TB.$$

As the dimension of $\mathcal{E}$ is one greater than the dimension of B, this is a line bundle. It is called the normal bundle as at each point $x \in B$, the fibre over this point is naturally identified with the tangent space at the marked point $s(x)$ to the elliptic curve $(\pi^{-1}(x), s(x))$ lying over x.

Since we may identify the fibre $(N_{\mathcal{E}}B)_x$ of the normal bundle of s over $x \in B$ with the tangent space $T_{s(x)}\mathcal{E}_x$ of the fibre of the elliptic curve bundle $\mathcal{E}$ over x, and the fibre $(\mathcal{H}_1(\mathcal{E};\mathbb{Z}))_x$ of the relative homology bundle over x with the first homology group $H_1(\mathcal{E}_x;\mathbb{Z})$ of the fibre of the elliptic curve bundle $\mathcal{E}$ over x, there exists a map

$$\Phi_{\mathcal{E}} : \mathcal{H}_1(\mathcal{E};\mathbb{Z}) \longrightarrow N_{\mathcal{E}}B$$

defined by the period map on each fibre. We now claim the following.

Proposition 2.16. *Let $(\mathcal{E}, B, \pi, s)$ be a family of elliptic curves over a complex manifold B. Then $(\mathcal{H}_1(\mathcal{E};\mathbb{Z}), N_{\mathcal{E}}B, \Phi_{\mathcal{E}})$ is a family of lattices over B.*

We have shown that the bundles are of the required form. It remains to show that $\Phi_{\mathcal{E}}$ is holomorphic. Since the map $\Phi_{\mathcal{E}}$ is constructed from integration against holomorphic differentials on each fibre, we are required to show that we can choose these holomorphic differentials to vary holomorphically as we move across fibres. As any nonzero holomorphic 1-form is a basis for all holomorphic 1-forms on an elliptic curve, and as holomorphicity is a local property, it suffices to show that about each $x \in B$ there exists an open neighbourhood U and a holomorphic 1-form ω_U on the fibre $\mathcal{E}_U$ over U, defined modulo 1-forms that vanish on the fibres, that is nonzero when restricted to each fibre.

This is, to the best of my understanding, a nontrivial result. It may be proved using methods in Hodge theory, and is a consequence of Voisin [31, §10.1 Theorem 10.10]. Taking this result as true, it may then be checked that this construction defines a map between isomorphism classes of families of elliptic curves and those of lattices, and that this map is in fact an inverse map to the one above.

It follows from this that the moduli problems for lattices and elliptic curves are equivalent.

Chapter 3

The Teichmüller Approach

In this chapter we return to the moduli problem for lattices. Now, safe in the knowledge that we are secretly dealing with the moduli space of elliptic curves, we attempt to squeeze as much information as we can out of it. Our first step is consider based lattices. These are related to lattices, but more happily have a solvable moduli problem. We next determine when two distinct based lattices are isomorphic as lattices. This information gives us a so-called 'coarse moduli space' for lattices—a space whose points parametrise isomorphism classes of lattices, and do so in a sensible way, but without a universal family.

3.1 A Solvable Moduli Problem

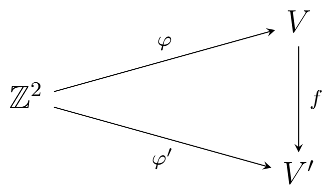
For all this talk of moduli spaces, we have not yet seen an example of one. We now rectify this.

In Chapter 1 we showed that an obstruction to the existence of a moduli space for lattices was the presence of nontrivial automorphisms of lattices. To construct a related solvable moduli problem then, we might try removing automorphisms. We do this by adding some extra structure—an ordered basis—to our lattices, and specifying that morphisms of lattices must now preserve this basis. This structure is sufficiently rigid to remove all automorphisms, and consequently the moduli problem for these based lattices becomes solvable. The cost of this approach, however, is that we also remove isomorphisms between objects, and so our number of isomorphism classes grows, leading to multiple points in our moduli space of based lattices representing the same isomorphism class of lattices. This can be, to some extent, controlled, but for now we pay no heed to this.

Definitions 3.1. A *based lattice* (V, φ) is a one-dimensional complex vector space

V together with a group homomorphism $\varphi : \mathbb{Z}^2 \hookrightarrow V$ such that the image of φ spans V as a real vector space.

Given two based lattices (V, φ) and (V', φ'), a *morphism of based lattices* is a complex linear map $f : V \to V'$ such that



commutes.

Observe that any morphism of based lattices is a nonzero complex linear map of 1-dimensional complex vector spaces, and hence an isomorphism.

It may seem that $\mathbb{Z}^2$ is simply the free abelian group on two generators, and so a based lattice is not any more, nor less, than a lattice. The difference between $\mathbb{Z}^2$ and an arbitrary rank two free abelian group Λ is that $\mathbb{Z}^2$ comes equipped with a canonical basis: $(1, 0)$ and $(0, 1)$. In the same way, we may think of the difference between a lattice and a based lattice to be the fact that a based lattice comes equipped with two special elements of V, given by $\varphi(1, 0)$ and $\varphi(0, 1)$. We shall call these two elements the *basis* for our lattice. A based lattice, then, can be viewed as a lattice for which we have selected two points in the image that generate the lattice as an additive group. A morphism of based lattices is a morphism of lattices that keeps track of this pair of points.

Just as the categories of lattices and elliptic curves are equivalent, it can be shown that the categories of based lattices and of elliptic curves with a basis for the first homology group are equivalent. The related moduli problems are also equivalent.

We begin our search for a moduli space by classifying all based lattices up to isomorphism.

Proposition 3.2. *Every based lattice (V, φ) is isomorphic to a unique based lattice of the form $(\mathbb{C}, \varphi_\tau)$,*

$$\varphi_\tau : \mathbb{Z}^2 \longrightarrow \mathbb{C};$$
$$(1, 0) \longmapsto \tau$$
$$(0, 1) \longmapsto 1,$$

where $\tau \in \mathbb{C} \setminus \mathbb{R}$. *In fact,* τ *is the unique complex number such that* $\varphi(1,0) = \tau\varphi(0,1)$.

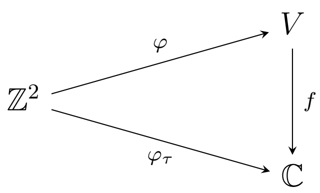
Proof. Let (V, φ) be a based lattice. We first show the existence of such a τ.

Since V is a complex vector space of dimension 1, any nonzero element of V forms a basis for V. In particular, $\varphi(1,0) \in V$ gives a basis. Thus there exists a unique element $\tau \in \mathbb{C}$ such that $\varphi(1,0) = \tau\varphi(0,1)$. Furthermore, $\varphi(1,0)$ and $\varphi(0,1)$ span V as a two-dimensional real vector space. Thus they are not $\mathbb{R}$-linearly independent, and so $\tau \notin \mathbb{R}$. This shows that $\tau \in \mathbb{C} \setminus \mathbb{R}$.

We now check that (V, φ) and $(\mathbb{C}, \varphi_\tau)$ are indeed isomorphic. Define the complex linear map $f : V \to \mathbb{C}$ such that $f(\varphi(0,1)) = 1$. This is an isomorphism of vector spaces. Since

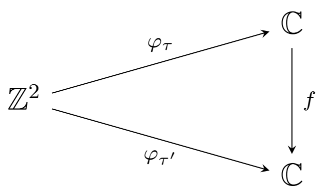
$$f(\varphi(1,0)) = f(\tau\varphi(0,1)) = \tau f(\varphi(0,1)) = \tau \cdot 1 = \tau = \varphi_\tau(1,0),$$

the diagram



commutes, and so $f : (V, \varphi) \to (\mathbb{C}, \varphi_\tau)$ is an isomorphism of based lattices.

It remains to show that τ is unique. Suppose that we have $\tau, \tau' \in \mathbb{C} \setminus \mathbb{R}$ such that $(\mathbb{C}, \varphi_\tau)$ and $(\mathbb{C}, \varphi_{\tau'})$ are isomorphic. It suffices to show that this implies $\tau = \tau'$. Since $(\mathbb{C}, \varphi_\tau)$ and $(\mathbb{C}, \varphi_{\tau'})$ are isomorphic, there exists a vector space isomorphism $f : \mathbb{C} \to \mathbb{C}$ such that



commutes. This implies that

$$f(1) = f(\varphi_\tau(0,1)) = \varphi_{\tau'}(0,1) = 1,$$

so f is the identity map. Thus

$$\tau = f(\tau) = f(\varphi_\tau(1,0)) = \varphi_{\tau'}(1,0) = \tau',$$

as desired. □

This shows that we have a bijection

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of based lattices}\end{array}\right\} \longleftrightarrow \mathbb{C}\setminus\mathbb{R}.$$

Observe that we have constructed the map between these two sets in a somewhat sensible way, taking the ratio of two basis elements of the lattice. Due to this, the element τ in $\mathbb{C}\setminus\mathbb{R}$ varies analytically as we vary the image in V of our basis elements analytically, and hence as we vary our based lattice analytically (whatever this might mean). This suggests $\mathbb{C}\setminus\mathbb{R}$ as a candidate fine moduli space for based lattices. Indeed this is so.

To build a family of lattices, we start with a Λ-fibred bundle—a bundle with fibres Λ and structure group $GL(2,\mathbb{Z})$. This is the group of automorphisms of Λ. Analogously, we wish to construct a family of based lattices from a $\mathbb{Z}^2$-fibred bundle with structure group equal to the group of automorphisms of $\mathbb{Z}^2$. But, as we have stressed, the key feature of $\mathbb{Z}^2$ in this context is its canonical basis, and hence we only wish to consider automorphisms that preserve this basis. The only such automorphism is the identity automorphism. Thus we are left with $\mathbb{Z}^2$-fibred bundles with the trivial structure group, and hence the trivial bundles $\mathbb{Z}^2\times B$. This motivates the following definition.

Definition 3.3. A *(holomorphic) family of based lattices* $(\mathcal{V},B,\Phi)$ over a complex manifold B consists of a line bundle $\mathcal{V}$ over B, and a holomorphic group bundle morphism $\Phi:\mathbb{Z}^2\times B\to\mathcal{V}$ such that for each $x\in B$ the pair $(\mathcal{V}_x,\Phi|_{\mathbb{Z}^2\times x})$ is a based lattice.

A *morphism of families of based lattices* is a morphism of line bundles that commutes with the lattice map. That is, given families of based lattices $(\mathcal{V}_1,B,\Phi_1)$ and $(\mathcal{V}_2,B,\Phi_2)$, a morphism of families of based lattices is a holomorphic map $F:\mathcal{V}_1\to\mathcal{V}_2$ of line bundles such that

$$\begin{array}{ccc} & \overset{\Phi_1}{\nearrow} & \mathcal{V}_1\\ \mathbb{Z}^2\times B & & \Big\downarrow F\\ & \underset{\Phi_2}{\searrow} & \mathcal{V}_2,\end{array}$$

commutes

In particular, this implies that the basis elements of the based lattice fibred over x varies analytically with x. Just as a based lattice is a lattice with two special points, a family of based lattices is a family of lattices with two special sections. Observe that any family of lattices is locally a family of based lattices: around any point in the base space we may take a trivialising neighbourhood of the Λ-fibred bundle, and hence choosing any isomorphism $\Lambda \to \mathbb{Z}^2$ gives the restriction of the family to this neighbourhood a basis.

The moduli problem for based lattices is then to represent the functor taking a complex manifold to the set of families of based lattices over it, and taking a holomorphic map of complex manifolds to the pullback map between sets of families. We shall solve this problem by exhibiting a universal family.

The purpose of fixing a basis is to make things more rigid. The next lemma shows we are succeeding.

Lemma 3.4. *Let $(\mathcal{V}, B, \Phi)$ be a family of based lattices over B. Then $\mathcal{V}$ is isomorphic to the trivial line bundle $\mathbb{C} \times B$.*

Proof. Define the map of vector bundles

$$\begin{aligned} F : \mathbb{C} \times B &\longrightarrow \mathcal{V}; \\ (z, x) &\longmapsto z\Phi\big((0,1), x\big). \end{aligned}$$

I claim that this is an isomorphism of vector bundles. Observe that the map $s(x) = \Phi\big((0,1), x\big)$ gives a global holomorphic section of the vector bundle $\mathcal{V}$, and that this section is nowhere zero as for all $x \in B$ its image $\Phi\big((0,1), x\big)$ gives a basis vector for the fibre $\mathcal{V}_x$. Since $\mathbb{C} \times B$ and $\mathcal{V}$ are line bundles, this ensures that F is a bijection. Observe further that any section of a holomorphic vector bundle is biholomorphic onto its image, as a holomorphic inverse is given by the bundle projection. Thus the map F, the complex linear extension of this section, is a biholomorphic bundle map, and hence an isomorphism of vector bundles. □

The most important example of a family of based lattices is the following.

Example 3.5 (The universal family of based lattices). We may define a family $(\mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}), \mathbb{C} \setminus \mathbb{R}, \Omega)$ of based lattices over $\mathbb{C} \setminus \mathbb{R}$ by

$$\begin{aligned} \Omega : \mathbb{Z}^2 \times (\mathbb{C} \setminus \mathbb{R}) &\longrightarrow \mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}); \\ \big((m, n), \tau\big) &\longmapsto (m\tau + n, \tau). \end{aligned}$$

Since $\Phi\big((1,0), \tau\big) = \tau$, $\Phi\big((0,1), \tau\big) = 1$, and $\tau \in \mathbb{C} \setminus \mathbb{R}$, the image of each fibre spans the vector space $\mathbb{C}_\tau$ as a 2-dimensional real vector space. We call this the universal family as every family of based lattices arises as a pullback of this family: the rest of this section will be devoted to proving this fact.

Let now $(\mathcal{V}, B, \Phi)$ be any family of based lattices. This induces a map

$$T_\Phi : B \longrightarrow \mathbb{C} \setminus \mathbb{R};$$
$$x \longmapsto \tau_x,$$

where τ_x is the unique element of $\mathbb{C} \setminus \mathbb{R}$ such that $\Phi((1,0),x) = \tau_x \Phi((0,1),x)$, and this map is holomorphic as Φ is holomorphic and $\Phi((0,1),x)$ is nonzero for all $x \in B$.

Conversely, suppose we have a holomorphic map $T : B \to \mathbb{C} \setminus \mathbb{R}$. We may define the pullback family of T (or more precisely, the pullback of the universal family along T) by letting Φ_T be the unique group bundle map such that

$$\Phi_T : \mathbb{Z}^2 \times B \longrightarrow \mathbb{C} \times B;$$
$$\big((1,0),x\big) \longmapsto (T(x),x),$$
$$\big((0,1),x\big) \longmapsto (1,x).$$

This is indeed a family as T is holomorphic and each $T(x)$ lies in $\mathbb{C} \setminus \mathbb{R}$. We call this the pullback of the universal family as Φ_T maps the fibre over $x \in B$ in the same way that Ω maps the fibre over $T(x) \in \mathbb{C} \setminus \mathbb{R}$. These two constructions lead to the following proposition.

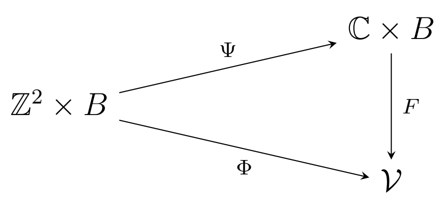
Proposition 3.6. *Let $(\mathcal{V}, B, \Phi)$ be any family of based lattices, and let T_Φ be the induced map $B \to \mathbb{C} \setminus \mathbb{R}$. Then the pullback family of T_Φ is isomorphic to $(\mathcal{V}, B, \Phi)$ as a family of based lattices via the map F constructed in the proof of Lemma 3.4.*

Proof. The pullback family of T_Φ is the family $(\mathbb{C} \times B, B, \Psi)$ defined by

$$\Psi : \mathbb{Z}^2 \times B \longrightarrow \mathbb{C} \times B;$$
$$\big((1,0),x\big) \longmapsto (\tau_x,x),$$
$$\big((0,1),x\big) \longmapsto (1,x),$$

where τ_x is the unique element of $\mathbb{C} \setminus \mathbb{R}$ such that $\Phi((1,0),x) = \tau_x \Phi((0,1),x)$.

We must show that



commutes.

This is true. For all $x \in B$ and all $(m, n) \in \mathbb{Z}^2$:

$$\begin{aligned} F \circ \Psi\big((m,n),x\big) &= F\Big(m\Psi\big((1,0),x\big) + n\Psi\big((0,1),x\big)\Big) \\ &= F\big(m\tau_x(1,x) + n(1,x)\big) \\ &= F\big(m\tau_x + n, x\big) \\ &= \Phi\big((m,n),x\big). \end{aligned}$$

□

Thus every family of based lattices can naturally be realised as a pullback of the universal family, and from this it follows that $\mathbb{C} \setminus \mathbb{R}$ is a fine moduli space for based lattices. In particular, this gives a complete classification of families of based lattices over any complex manifold B via the bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of families} \\ \text{of based lattices over } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{holomorphic maps} \\ B \longrightarrow \mathbb{C} \setminus \mathbb{R} \end{array} \right\}.$$

3.2 Lattices and Based Lattices

The space $\mathbb{C} \setminus \mathbb{R}$ is a moduli space for based lattices, and hence its points are in bijection with isomorphism classes of based lattices. While it is not possible to use the universal family for based lattices to construct a universal family for lattices—indeed, no such family exists—it is possible to at least construct a space classifying all individual lattices. To do this we need to know when two based lattices are isomorphic as lattices. A convenient language for expressing this is the language of group actions.

The action of $\mathrm{GL}(2, \mathbb{Z})$ on $\mathbb{C} \setminus \mathbb{R}$

With the notation as in the previous section, let $(\mathbb{C}, \varphi_\tau)$, $(\mathbb{C}, \varphi_{\tau'})$ be based lattices, where $\tau, \tau' \in \mathbb{C} \setminus \mathbb{R}$. Suppose that they are isomorphic as lattices. Then there exists a complex linear map $f_V : \mathbb{C} \to \mathbb{C}$ and a group isomorphism $f_\Lambda : \mathbb{Z}^2 \to \mathbb{Z}^2$ such that

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{\varphi_\tau} & \mathbb{C} \\ {\scriptstyle f_\Lambda}\downarrow & & \downarrow{\scriptstyle f_V} \\ \mathbb{Z}^2 & \xrightarrow{\varphi_{\tau'}} & \mathbb{C} \end{array}$$

Now, as an automorphism of $\mathbb{Z}^2$, f_Λ can be represented by multiplication by an element of $\mathrm{GL}(2, \mathbb{Z})$. For notational reasons that will become apparent, let us represent it by the matrix $A^{-1} := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \in \mathrm{GL}(2,\mathbb{Z})$ acting on $\mathbb{Z}^2$, considered as row vectors, by multiplication on the right. More explicitly $A^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. The commutativity of the above square implies that

$$f(\tau) = f(\varphi(1,0)) = \varphi_{\tau'}(f_\Lambda(1,0)) = \varphi_{\tau'}((1,0)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)) = \varphi_{\tau'}(d,-b) = d\tau' - b,$$

$$f(1) = f(\varphi(0,1)) = \varphi_{\tau'}(f_\Lambda(0,1)) = \varphi_{\tau'}((0,1)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)) = \varphi_{\tau'}(-c,a) = -c\tau' + a.$$

Dividing the first line by the second, the $\mathbb{C}$-linearity of the vector space map f thus shows that $\tau = \frac{d\tau'-b}{-c\tau'+a}$ or, equivalently, that $\tau' = \frac{a\tau+b}{c\tau+d}$.

Conversely, suppose that there exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$ such that $\tau' = \frac{a\tau+b}{c\tau+d}$. Then, defining the group homomorphisms

$$\begin{aligned} f_V^{A,\tau} :\ \mathbb{C} &\longrightarrow \mathbb{C}; \\ z &\longmapsto \frac{z}{c\tau+d} \\ f_\Lambda^A : \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2; \\ (m,n) &\longmapsto (m,n)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} = (dm-bn, -cm+an), \end{aligned}$$

it is easy to check that the square

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{\varphi_\tau} & \mathbb{C} \\ {\scriptstyle f_\Lambda^A}\downarrow & & \downarrow{\scriptstyle f_V^A} \\ \mathbb{Z}^2 & \xrightarrow{\varphi_{\tau'}} & \mathbb{C} \end{array}$$

commutes and hence that show that the lattices $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau)$ and $(\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'})$ are isomorphic.

We thus have the theorem:

Theorem 3.7. *Let $\tau, \tau' \in \mathbb{C} \setminus \mathbb{R}$. Then the map $A \mapsto (f_V^{A,\tau}, f_\Lambda^A)$ gives a bijection*

$$\left\{ A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z}) \,\middle|\, \tau' = \tfrac{a\tau+b}{c\tau+d} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{isomorphisms of the lattices} \\ (\mathbb{C}, \mathbb{Z}^2, \varphi_\tau) \textit{ and } (\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'}) \end{array} \right\}.$$

Motivated by this, we wish to define an action of $\mathrm{GL}(2,\mathbb{Z})$ on $\mathbb{C} \setminus \mathbb{R}$ by having each $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$ induce the map:

$$\begin{aligned} A : \mathbb{C} \setminus \mathbb{R} &\longrightarrow \mathbb{C} \setminus \mathbb{R}; \\ \tau &\longmapsto \frac{a\tau+b}{c\tau+d}. \end{aligned}$$

To check that this map is well-defined the following lemma, although simple, is useful.

Lemma 3.8. *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$. *Then*

$$\operatorname{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\det A}{|c\tau + d|^2}\operatorname{Im}(\tau).$$

Proof. Let $\tau = r + is \in \mathbb{C} \setminus \mathbb{R}$, where $r, s \in \mathbb{R}$. Then:

$$\frac{a\tau + b}{c\tau + d} = \frac{(a\tau + b)(c\overline{\tau} + d)}{|c\tau + d|^2} = \frac{(ac|\tau|^2 + bd + (ad + bc)r) + i(ad - bc)s}{|c\tau + d|^2}.$$

The lemma follows from this expression. □

As A lies in $\mathrm{GL}(2,\mathbb{Z})$, the integers c and d are not both zero. Since $\tau \in \mathbb{C} \setminus \mathbb{R}$, we thus cannot have $c\tau + d = 0$. This implies that $\frac{a\tau+b}{c\tau+d}$ is a well-defined complex number. This also implies that $\frac{\det A}{|c\tau+d|^2}$ is a nonzero real number, and so together with the lemma implies that $\frac{a\tau+b}{c\tau+d}$ is non real when τ is non real. Thus for each A we have a well-defined holomorphic map. It is a short computation to check that for all $A, B \in \mathrm{GL}(2,\mathbb{Z})$ we have $(AB)\tau = A(B\tau)$, and so this is indeed an action.

Oriented based lattices

The moduli space for based lattices, $\mathbb{C} \setminus \mathbb{R}$, consists of two disjoint connected components. It will be simpler for us to deal with just one of these components. In doing so, we do not lose any information.

The existence of two disjoint components reflects the fact that families of based lattices over a connected base space can be separated into two distinct classes. Let B be a connected space, and let $(\mathcal{V}, B, \Phi)$ be a family of based lattices. For $x \in B$, let $\theta(x)$ be the counterclockwise angle, expressed in radians in the interval $(-\pi, \pi]$, between the basis vectors $\Phi((0,1), x)$ and $\Phi((1,0), x)$ of the lattice fibred over x. (More formally, this is the argument of the complex number τ_x such that $\Phi((1,0), x) = \tau_x \Phi((0,1), x)$.) Noting that these basis vectors are $\mathbb{R}$-linearly independent, the angle between them cannot be 0 or π, and so we see that $\theta(x)$ takes values in the disjoint set $(-\pi, 0) \cup (0, \pi)$. But as a function of x, θ must be continuous for any continuous family over B. Hence for any such family, θ either takes values in the interval $(-\pi, 0)$ or the interval $(0, \pi)$, but not both. We shall call families of the first type families with *negative orientation*, and families of the second type families with *positive orientation*. This is a continuous invariant on the set of based lattices. This is a general phenomenon: connected components of the moduli space correspond to deformation invariants of objects.

Observe that a family $(\mathcal{V}, B, \Phi)$ of positive orientation can be transformed into a family of negative orientation by fibrewise precomposition of the map $\mathbb{Z}^2 \to$

$\mathbb{Z}^2; (m, n) \mapsto (n, m)$ with Φ, and vice versa. In order to understand all families of based lattices, it thus suffices only to study families with one orientation. We thus restrict our attention to lattices with a basis of positive orientation. To ease terminology slightly, we will call such based lattices *oriented based lattices.* More formally:

Definition 3.9. Let (V, φ) be a based lattice. We call (V, φ) an *oriented based lattice* if there exists a complex number τ with $\operatorname{Im} \tau > 0$ such that $\varphi(1, 0) = \tau\varphi(0, 1)$. A family of based lattices $(\mathcal{V}, B, \Phi)$ is a *family of oriented based lattices* if for each $x \in B$ the fibre $(\mathcal{V}_x, \Phi_x)$ over x is an oriented based lattice.

It follows from the above discussion that the upper half plane

$$\mathfrak{h} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

is a moduli space for such families. Furthermore, by Lemma 3.8, we see that $A \in \mathrm{GL}(2, \mathbb{Z})$ maps $\tau \in \mathfrak{h}$ to another element of $\mathfrak{h}$ if and only if A has positive determinant. Thus we restrict the action of $\mathrm{GL}(2, \mathbb{Z})$ on $\mathbb{C} \setminus \mathbb{R}$ to an action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathfrak{h}$, and to describe when two oriented based lattices are isomorphic as lattices we have the analogue of Theorem 3.7:

Theorem 3.10. *Let $\tau, \tau' \in \mathfrak{h}$. Then the map $A \mapsto (f_V^{A,\tau}, f_\Lambda^A)$ gives a bijection*

$$\left\{ A \in \mathrm{SL}(2, \mathbb{Z}) \,\middle|\, A\tau = \tau' \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{isomorphisms of the lattices} \\ (\mathbb{C}, \mathbb{Z}^2, \varphi_\tau) \textit{ and } (\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'}) \end{array} \right\}.$$

Another way to think of our choice of orientation is that we have taken the quotient of $\mathbb{C}\backslash\mathbb{R}$ by the order two group generated by the element $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{GL}(2, \mathbb{Z})$ that swaps $(1, 0)$ and $(0, 1)$. A quotient by this group forgets the ordering we have on our basis, and so if we identify based lattices up to the action of this group, our objects are in fact based lattices with an unordered basis. This quotient identifies $\tau \in \mathbb{C} \setminus \mathbb{R}$ with $1/\tau$. Since $\operatorname{Im} \tau > 0$ if and only if $\operatorname{Im}(1/\tau) < 0$, we may choose representatives in the upper half plane. Similarly, since we may write $\mathrm{GL}(2, \mathbb{Z})$ as the semidirect product $\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \rtimes \mathrm{SL}(2, \mathbb{Z})$ given by the action

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix},$$

we may choose representatives of $\mathrm{GL}(2, \mathbb{Z})/\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ that are precisely $\mathrm{SL}(2, \mathbb{Z})$. Thus studying families of oriented based lattices is equivalent to studying families of lattices with an unordered basis.

The Space $\mathcal{M}$

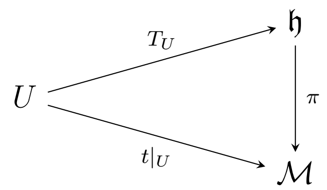
The above discussion shows that the space

$$\mathcal{M} := \mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h} \cong \mathrm{GL}(2,\mathbb{Z})\backslash(\mathbb{C}\setminus\mathbb{R})$$

uniquely parametrises isomorphism classes of lattices (and hence elliptic curves too). At the moment, however, we only know this construction holds meaning at the level of sets; it says nothing about the modulation of lattices and hence, equivalently, nothing about families of lattices.

Since the complex structure on $\mathfrak{h}$ captures the way based lattices modulate, however, we might also suspect that $\mathcal{M}$ parametrises lattices in a sensible way, capturing some, if not all, of the ways lattices can vary. More precisely, we mean the following: let $(\mathcal{V},\mathcal{E},\Phi)$ be a family of lattices over a complex manifold B. For each x in B, there exists a unique point $t(x)\in\mathcal{M}$ such that the lattice $(\mathcal{V}_x,\mathcal{E}_x,\Phi|_{\mathcal{E}_x})$ over x belongs to the isomorphism class of lattices parametrised by $t(x)$. We may thus define a map $t : B \to \mathcal{M}$ mapping x to $t(x)$. For the complex structure on $\mathcal{M}$ to capture what we want, we wish that for every family over any complex manifold B, the map $t : B\to\mathcal{M}$ is holomorphic. We call such a space a *coarse moduli space*, as it has the beginnings of the features of a moduli space. To distinguish this from a moduli space proper, we will when necessary call the latter a *fine moduli space*.

For $\mathcal{M}$ to be a coarse moduli space for lattices, it is in the first instance necessary for $\mathcal{M}$ to have a complex structure. Suppose that this is so, and that the quotient map $\pi : \mathfrak{h}\to\mathcal{M}$ is holomorphic. Let $(\mathcal{V},\mathcal{E},\Phi)$ be a family of lattices over a complex manifold B, and let $t : B\to\mathcal{M}$ be the induced map. Given a point $x\in B$, let U be a neighbourhood of B such that $\mathcal{V}_U$ and $\mathcal{E}_U$ are the trivial bundles. Then, choosing some isomorphism $\Lambda\to\mathbb{Z}^2$, we may view $(\mathcal{V}_U,\mathcal{E}_U,\Phi|_{\mathcal{E}_U})$ as a family of based lattices over U. The fact that $\mathfrak{h}$ is a moduli space then induces a holomorphic map $T_U : U\to\mathfrak{h}$ such that



commutes. Since π is holomorphic by hypothesis, this shows that $t|_U$ is holomorphic. Furthermore, since this is true for every $x\in B$, this shows that t is holomorphic.

We can thus conclude that a sufficient condition for $\mathcal{M}$ to be a coarse moduli space for lattices is that it has a complex structure such that the quotient map $\pi : \mathfrak{h} \to \mathcal{M}$ is holomorphic. In the next section we give a general condition under which a complex structure on a Riemann surface descends in this way to a quotient by a group action.

3.3 Descent of Complex Structures

This section shows that the quotient of a Riemann surface by a properly discontinuous action is again naturally a Riemann surface. The results and arguments in this section are variously inspired by ideas in Milne [24, Ch.I §2], Miranda [26, Ch.III §3] and Hain [14, App. A].

In our preliminary notes on complex manifolds, we proved that when a group acts freely and properly discontinuously on a complex manifold, there exists a natural complex structure of the same dimension on the quotient such that the projection map is holomorphic. Unfortunately the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathfrak{h}$ is not free. Fortunately, in a single dimension this is not required.

Theorem 3.11. *Let X be a Riemann surface, and let G be a group acting properly discontinuously on X. Then $G\backslash X$ has a unique complex structure, of the same dimension as that on X, such that the projection $\pi : X \to G\backslash X$ is holomorphic.*

Without loss of generality, we may in addition take the hypothesis that G acts *effectively*—that is, that the only element that acts trivially on all of X is the identity. Recalling that an action of G on X defines a map from G to $\mathrm{Aut}(X)$, we note that if G does not act effectively, we may replace G with its image G' in $\mathrm{Aut}(X)$. It is clear that the G-orbits of X are the same as the G'-orbits of X, and so the respective quotient spaces are naturally homeomorphic.

Supposing then that G acts effectively, the subtlety in this theorem lies in the handling of the points that have nontrivial stabilisers. Our work is eased by the fact that there are not too many of these points.

Lemma 3.12. *Let X be a Riemann surface, and let G be a group acting properly discontinuously and effectively on X. Then the set of points of X with nontrivial stabilisers is discrete.*

Proof. We prove by contradiction. Let $\{x_i\}$ be a sequence of points in X, converging to a point $x \in X$, such that each x_i has nontrivial stabiliser. This implies that each x_i is fixed by some nonidentity $g_i \in G$. As the action of G is properly discontinuous, it is possible to choose a neighbourhood U of x such that the set $\{g \in G \mid gU \cap U \neq \varnothing\}$ is finite. Thus, passing to a subsequence of $\{x_i\}$ contained inside U, there are only finitely many distinct g_i. Moreover, since there are only finitely many distinct g_i, we may once again pass to a subsequence and assume that all these g_i are the same element g of G.

We now have an element g of G, not the identity, that fixes a convergent sequence of points $\{x_i\}$. Since g acts analytically it must hence also fix the limit point x. But by the identity theorem, this implies that g acts trivially in a neighbourhood of x, and hence on all of X. This contradicts the hypothesis that G acts effectively. □

Furthermore, the stabilisers of each of these points are not complicated: they are necessarily finite cyclic groups.

Lemma 3.13. *Let X be a Riemann surface and let G be a group acting properly discontinuously and effectively on X. Then, given any point $x \in X$, the stabiliser $\mathrm{Stab}_G(x)$ of x is a finite cyclic group.*

Proof. Observe that for any neighbourhood U of x, the stabiliser of x is necessarily contained in the set $\{g \in G \mid gU \cap U \neq \varnothing\}$. Since G acts properly discontinuously, we may conclude that the stabiliser of x is finite. It remains to show that it is cyclic. We do this by showing that it injects into the multiplicative group $\mathbb{C}^\times$.

Let (U, z) be a coordinate neighbourhood of x such that $z(x) = 0 \in \mathbb{C}$. We shall work in this local coordinate. Let g be an element of $\mathrm{Stab}_G(x)$. Since g acts analytically and maps x to x, we may write its action in this local coordinate as the holomorphic function

$$f_g(z) = \sum_{n=1}^{\infty} a_{gn} z^n.$$

Note that there is no constant term, as f_g fixes 0. Furthermore, as g induces an automorphism of X, f_g is locally a biholomorphism, and so $a_{g1} \neq 0$. Consider now the map

$$\begin{aligned} \rho : \mathrm{Stab}_G(x) &\longrightarrow \mathbb{C}^\times; \\ g &\longmapsto f_g'(0) = a_{g1}. \end{aligned}$$

This is a homomorphism of groups as $f_{g_2 g_1} = f_{g_2} \circ f_{g_1}$ and hence

$$\rho(g_2 g_1) = f'_{g_2 g_1}(0) = f'_{g_2}(f_{g_1}(0)) f'_{g_1}(0) = f'_{g_2}(0) f'_{g_1}(0) = \rho(g_2)\rho(g_1).$$

We next show that ρ is injective.

Suppose that g lies in the kernel of ρ. Then $a_{g1} = 1$. We wish to show that $a_{gn} = 0$ for all $n \neq 1$. For the sake of contradiction suppose otherwise. Then there exists $n \geq 2$ such that

$$f_g(z) \equiv z + a_{gn} z^n \mod z^{n+1},$$

where $a_{gn} \neq 0$. Observe now that for any natural number k, we have

$$\begin{aligned} f_g(z + k a_{gn} z^n) &\equiv (z + k a_{gn} z^n) + a_{gn}(z + k a_{gn} z^n)^n \\ &\equiv z + k a_{gn} z^n + a_{gn} z^n \\ &\equiv z + (k+1) a_{gn} z^n \end{aligned}$$

when working mod z^{n+1}. By induction this implies that

$$f_{g^k}(z) \equiv z + k a_{gn} z^n \mod z^{n+1}.$$

Taking k to be the order of g, which is finite as $\mathrm{Stab}_G(x)$ is finite, we see that $k a_{gn} = 0$. Since k is nonzero, this implies that $a_{gn} = 0$, a contradiction. Thus for any g in the kernel of ρ, $f_g(z) = z$. By the identity theorem, implies that g acts trivially. Since G acts effectively, this shows g must be the identity, so the kernel of ρ is trivial and ρ is injective.

We have now shown that $\mathrm{Stab}_G(x)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$. Since all such subgroups are cyclic, this proves the lemma. □

We next identify the neighbourhoods we shall use to define charts on the quotient. These are the analogues of those used in the case that the action is free.

Lemma 3.14. *Let X be a Riemann surface, and let G be a discrete group acting properly discontinuously on X. Then, for any two points $x, y \in X$, there exist open neighbourhoods U of x and V of y such that $gU \cap V = \varnothing$ for every $g \in G$ such that $gx \neq y$.*

Proof. Since the action of G is properly discontinuous, we may choose open neighbourhoods U of x and V of y such that $\{g \in G \mid gU \cap V \neq \varnothing\}$ is finite. Suppose $h \in G$ is a member of this set, but does not map x to y. We will show we can shrink the neighbourhoods U and V to exclude h from this set.

Since X is Hausdorff, y and hx are distinct points of X, and h induces an automorphism of X, there exist open neighbourhoods U' of x and V' of y such that $hU' \cap V' = \varnothing$. Taking $U'' = U \cap U'$ and $V'' = V \cap V'$, we have neighbourhoods of x

and y such that $h \notin \{g \in G \mid gU'' \cap V'' \neq \varnothing\} \subsetneq \{g \in G \mid gU \cap V \neq \varnothing\}$. Iterating this process finitely many times to exclude all such h, we may thus suppose, without loss of generality, that we have chosen U and V such that $gU \cap V = \varnothing$ when $gx \neq y$. □

Lemma 3.15. *Let X be a Riemann surface, and let G be a group acting properly discontinuously on X. Then, for each point $x \in X$, there exists an open neighbourhood U of x such that $gU \cap U = \varnothing$ for every $g \notin \mathrm{Stab}_G(x)$ and $\mathrm{Stab}_G(x)$ acts freely on $U \setminus \{x\}$.*

Proof. Taking both points to be x in Lemma 3.14, we have a neighbourhood U_0 of x such that $\{g \in G \mid gU_0 \cap U_0 \neq \varnothing\}$ is the stabiliser of x in G. Since, by Lemma 3.12, the set of points with nontrivial stabilisers is discrete, we may also take a neighbourhood U_1 of x containing no points, other than perhaps x, with nontrivial stabiliser. Take now

$$U = \bigcap_{g \in \mathrm{Stab}_G(x)} g(U_0 \cap U_1).$$

Observe that this intersection is finite by Lemma 3.13, so this set is indeed open. Since $U \subset U_0$, we have $gU \cap U = \varnothing$ for all $g \notin \mathrm{Stab}_G(x)$. Furthermore, by construction it is clear that U and hence $U \setminus \{x\}$ are invariant under $\mathrm{Stab}_G(x)$, and that the subsequent action of $\mathrm{Stab}_G(x)$ on $U \setminus \{x\}$ is free. □

The above lemmas indicate that the quotient map induced by a properly discontinuous action looks, around each point, a little like $z \mapsto z^k$ for some varying natural number k. We use this idea to give the quotient a complex structure—the main theorem then follows.

Proof of Theorem 3.11. It is an immediate corollary of Lemma 3.14 that the quotient $G\backslash X$ is Hausdorff.

We turn our attention to defining a complex structure on $G\backslash X$. Recall that we have a quotient map $\pi : X \to G\backslash X$. Let $x \in G\backslash X$, and let $\widetilde{x} \in \pi^{-1}(x)$. Let $\widetilde{U}$ be a neighbourhood of $\widetilde{x}$ given by Lemma 3.15. Shrinking $\widetilde{U}$ if necessary, let $\widetilde{z}$ be a local coordinate on $\widetilde{U}$, centred at $\widetilde{x}$, and for each $g \in \mathrm{Stab}_G(\widetilde{x})$ let $f_g : \widetilde{U} \to \mathbb{C}$ be the holomorphic map induced in this coordinate by the action of g. Define the map

$$\begin{aligned} \widetilde{f} : \widetilde{U} &\longrightarrow \mathbb{C}; \\ z &\longmapsto \prod_{g \in \mathrm{Stab}_G(x)} f_g(z). \end{aligned}$$

By construction, $\pi(\widetilde{U}) \cong \mathrm{Stab}_G(x)\backslash\widetilde{U}$. Since $\widetilde{f}$ is invariant under $\mathrm{Stab}_G(\widetilde{x})$, by the universal property of quotients it descends to a function $f : \pi(\widetilde{U}) \to \mathbb{C}$ such that

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\ \widetilde{f}\ } & \mathbb{C} \\ \big\downarrow & \nearrow_{f} & \\ \mathrm{Stab}_G(\widetilde{x})\backslash\widetilde{U} \cong \pi(\widetilde{U}) & & \end{array}$$

commutes. I now claim that $(\pi(\widetilde{U}), f)$ gives a coordinate neighbourhood of x.

Since $\widetilde{f}$ is continuous and open, f is also. Furthermore, since both $\widetilde{f}$ and π are m-to-1 about $\widetilde{x}$, where $m = \#\mathrm{Stab}_G(\widetilde{x})$, the induced function f is injective. This proves that f is a homeomorphism onto an open subset of $\mathbb{C}$, and hence does define a coordinate neighbourhood.

We have now defined a coordinate neighbourhood about each point in the quotient $G\backslash X$. To show that these coordinate neighbourhoods give an atlas for X, we now need to show that they are pairwise compatible. Let (U_1, f_1), (U_2, f_2) be coordinate neighbourhoods in $G\backslash X$ of the above form, descending from coordinate neighbourhoods $(\widetilde{U}_1, z_1)$ and $(\widetilde{U}_2, z_2)$ of X. As the set of points with nontrivial stabilisers is discrete, we may assume without loss of generality that (U_1, f_1) is constructed about a point with trivial stabiliser, and hence biholomorphic to $(\widetilde{U}_1, z_1)$ via π. In particular, this implies that $\widetilde{U}_1 \cap \widetilde{U}_2$ is biholomorphic to $U_1 \cap U_2$ via π. Observe that $\widetilde{f}_2$, the lift of f_2 to $\widetilde{U}_2$, is holomorphic with nowhere vanishing derivative on $\widetilde{U}_1 \cap \widetilde{U}_2$, as this set contains no points with nontrivial stabilisers. Thus, restricted to $f_1(U_1 \cap U_2)$, the composition $\widetilde{f}_2 \circ \pi^{-1} \circ f_1^{-1}$ is a composition of holomorphic functions with nowhere vanishing derivatives and so a biholomorphism onto its image. But $f_2 \circ f_1^{-1} = \widetilde{f}_2 \circ \pi^{-1} \circ f_1^{-1}$ on $f_1(U_1 \cap U_2)$, so the same is true of $f_2 \circ f_1^{-1}$. This shows that (U_1, f_1) and (U_2, f_2) are compatible. □

3.4 A Coarse Moduli Space

Having shown that a complex structure on a Riemann surface naturally descends to the quotient by a properly discontinuous action, we now wish to show that the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathfrak{h}$ is properly discontinuous. An action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathfrak{h}$ may be defined analogously to our action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathfrak{h}$, and it is a basic fact in the theory of modular forms that for any discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ with this action is properly discontinuous [24, Ch.I §2]. While this is not difficult to develop, we instead take a more explicit approach, computing a fundamental domain for the action. The details of the quotient map will be useful. The computation of the fundamental domain can be found in, among other sources, Serre [27, Ch.VII §1] and Silverman [29, Ch.I §1].

Our first task is to find a connected subset of $\mathfrak{h}$ containing exactly one point from each $\mathrm{SL}(2,\mathbb{Z})$-orbit. More specifically, we look for a fundamental domain for the action.

Definition 3.16. Let a group G act on a space X. A *fundamental domain* for the action of G on X is a connected open subset D of X such that

(i) for all $x \in X$ there exists $g \in G$ such that $gx \in \overline{D}$, where $\overline{D}$ is the closure of D in X, and

(ii) if $x, y \in D$ are distinct, then for all $g \in G$ we have $gx \neq y$.

The first condition then implies that a representative of every G-orbit of X lies in the closure of the fundamental domain, and the second implies that, within the fundamental domain, this representative is unique. Put another way, D is a fundamental domain if the canonical map $\overline{D} \to G\backslash X$ is surjective and the map $D \to G\backslash X$ is injective.

Define

$$\mathcal{D} := \{z \in \mathfrak{h} \mid |z| > 1, |\mathrm{Re}\, z| < \tfrac{1}{2}\}.$$

This domain can be pictured, as in Figure 3.1, as a hyperbolic triangle in the upper half plane with vertices $\rho := e^{2\pi i/3}$, $-\overline{\rho}$ and ∞.

Theorem 3.17. *The domain $\mathcal{D}$ is a fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathfrak{h}$.*

Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. In the course of proving Theorem 3.17, we shall also prove the following.

Proposition 3.18. *Let τ, τ' be distinct elements of $\overline{\mathcal{D}}$. Then there exists $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $A\tau = \tau'$ if and only if either*

1. $\mathrm{Re}\,\tau = \frac{1}{2}$, $\tau' = \tau - 1$ *and* $A = \pm T^{-1}$;

2. $\mathrm{Re}\,\tau = -\frac{1}{2}$, $\tau' = \tau + 1$ *and* $A = \pm T$; *or,*

3. $|\tau| = 1$, $\tau' = -\frac{1}{\tau}$ *and* $A = \pm S$.

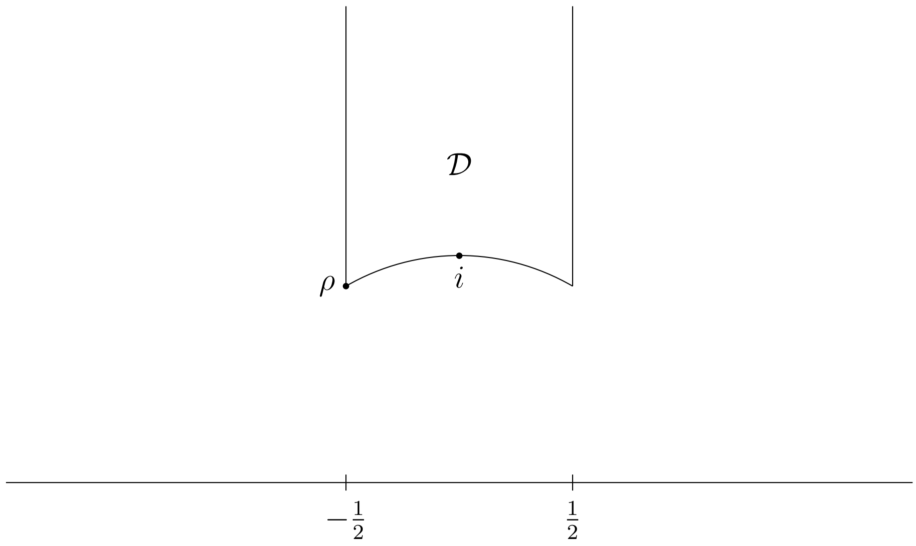


Figure 3.1: The domain $\mathcal{D}$ as a subset of $\mathfrak{h}$.

Proposition 3.19. *Let $\tau \in \overline{\mathcal{D}}$. Then*

$$\mathrm{Stab}(\tau) = \begin{cases} \langle S \rangle & \text{if } \tau = i; \\ \langle ST \rangle & \text{if } \tau = \rho; \\ \langle TS \rangle & \text{if } \tau = -\overline{\rho}; \\ \langle -1 \rangle & \text{otherwise.} \end{cases}$$

The former proposition shows that $\mathcal{M}$ may be constructed from $\overline{\mathcal{D}}$ by identifying points the boundary of $\mathcal{D}$ mapped to one another by reflection in the imaginary axis. This shows that the space $\mathcal{M}$ is simply connected. The latter proposition identifies the stabilisers—we shall use this information in discussing automorphisms of elliptic curves. Observe that the points ρ and $-\overline{\rho}$ are equivalent, as $T\rho = -\overline{\rho}$. A consequence is that the stabilisers of ρ and $-\overline{\rho}$ are isomorphic via conjugation by T.

Proof of Theorem 3.17 and Propositions 3.18 and 3.19. We first show that the closure of $\mathcal{D}$ contains an element of each $\mathrm{SL}(2,\mathbb{Z})$-orbit of $\mathfrak{h}$. Let $\tau \in \mathfrak{h}$. We wish to show that there exists $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $A\tau \in \overline{\mathcal{D}}$. Note first that given any $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, Lemma 3.8 gives that

$$\mathrm{Im}\,(B\tau) = \frac{1}{|c\tau + d|^2}\mathrm{Im}\,(\tau).$$

Since c and d are integers, $|c\tau + d|^2$ takes a (nonzero) minimum value, and hence we can choose B such that $\mathrm{Im}\,(B\tau)$ is maximal. Choose now an integer k such that $|\mathrm{Re}\,(B\tau) + k| \le \frac{1}{2}$. Let $A = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right) B\tau$. This is a product of two elements of $\mathrm{SL}(2,\mathbb{Z})$, and hence itself an element of $\mathrm{SL}(2,\mathbb{Z})$. I now claim that A is an element of $\mathrm{SL}(2,\mathbb{Z})$ such that $A\tau \in \overline{\mathcal{D}}$.

By Lemma 3.8, $\mathrm{Im}\,\big(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) A\tau\big) = \frac{1}{|A\tau|^2}\mathrm{Im}\,(A\tau)$. On the other hand, $\mathrm{Im}\,(A\tau) = \mathrm{Im}\,(B\tau)$ which was chosen to be maximal. Thus

$$\frac{1}{|A\tau|^2}\mathrm{Im}\,(A\tau) = \mathrm{Im}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) A\tau\right) \le \mathrm{Im}\,(A\tau).$$

This shows that $|A\tau| \ge 1$. Furthermore, by choice of k, we have $|\mathrm{Re}\,(A\tau)| = |\mathrm{Re}\,(B\tau) + k| \le \frac{1}{2}$. Thus $A\tau \in \overline{\mathcal{D}}$, as claimed.

To prove Theorem 3.17 it is necessary to show that $\mathcal{D}$ has the second property of a fundamental domain: that $\mathcal{D}$ contains at most one element from each orbit. In order to prove this, we investigate more generally when two elements of $\overline{\mathcal{D}}$ are related by an element of $\mathrm{SL}(2,\mathbb{Z})$. In doing so we will prove Propositions 3.18 and 3.19 too.

Suppose that we have $\tau, \tau' \in \overline{\mathcal{D}}$ such that $\tau' = A\tau$ for some $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2,\mathbb{Z})$. Without loss of generality we assume that $\mathrm{Im}\,(\tau') \ge \mathrm{Im}\,(\tau)$. Since, once again as an application of Lemma 3.8, we have $\mathrm{Im}\,(\tau') = \frac{1}{|c\tau+d|^2}\mathrm{Im}\,\tau$, this implies that $|c\tau + d|^2 \le 1$, and hence that $(c\mathrm{Im}\,(\tau))^2 \le 1$. But τ is in the set $\overline{\mathcal{D}}$, so $\mathrm{Im}\,(\tau) \ge \sqrt{\frac{3}{2}}$. We are restricted to three possible values for c: -1, 0 and 1. Since $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts trivially, we may take note that $-I$ lies in the stabiliser of every element of $\mathcal{D}$, and only consider the cases $c = 0$ and $c = 1$—the case $c = -1$ can be turned into the case $c = 1$ by composition with $-I$.

- Suppose that $c = 0$. Then $|d| \le 1$ and, since $ad - bc = 1$, in fact $d = a = \pm 1$. The matrix A thus acts by addition of $\pm b$. By inspection of $\mathcal{D}$, this then implies that $\mathrm{Re}\,(\tau) = \pm\frac{1}{2}$ and $\tau' = \tau \mp 1$.

- Suppose that $c = 1$. Then, since $|\tau + d|^2 \le 1$, we have

  $$(\mathrm{Re}\,(\tau) + d)^2 + \mathrm{Im}\,(\tau)^2 \le 1$$

  and so

  $$|\tau|^2 \le 1 - 2d\mathrm{Re}\,(\tau) - d^2 = 1 - d(2\mathrm{Re}\,(\tau) \pm 1) - d(d \mp 1).$$

  But d is an integer and $2\mathrm{Re}\,(\tau) \le 1$, so by some choice of sign we have both $d(2\mathrm{Re}\,(\tau) \pm 1) \ge 0$ and $d(d \mp 1) \ge 0$. Since $|\tau| \ge 1$, this implies that $|\tau| = 1$ and that $d = -1$, 0 or 1. We again split into cases:

– Suppose that $c = 1$, $d = 0$. Then $A = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and

$$A\tau = a - \tfrac{1}{\tau} = a - \overline{\tau}.$$

(Recall that $|\tau| = 1$, so $\frac{1}{\tau} = \overline{\tau}$.) Since τ and $A\tau$ are in $\overline{\mathcal{D}}$, this implies $|\mathrm{Re}\,(\tau)|$ and $|a - \mathrm{Re}\,(\tau)|$ are both not greater than $\frac{1}{2}$. Since $a \in \mathbb{Z}$, we may this conclude that $a = 0$ unless $\tau = \rho$ or $-\overline{\rho}$. In the former case we may take $a = -1$, so $A = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = (ST)^2$ and $A\rho = \rho$, and in the latter case we may take $a = 1$, so $A = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = TS$ and $A(-\overline{\rho}) = -\overline{\rho}$. Observe that when $a = 0$, we have $A = S$ and $A\tau = -\overline{\tau}$. Thus in this case A acts by reflection in the imaginary axis. In particular, the only fixed point of this is the fourth root of unity i.

– Suppose that $c = 1$, $d = -1$. In this case we have chosen d such that $d(d+1) = 0$, and hence $d(2\mathrm{Re}\,(\tau) - 1) = 0$. Thus $\mathrm{Re}\,(\tau) = \frac{1}{2}$, and so $\tau = -\overline{\rho}$. From the determinant relation $ad - bc = 0$, we also determine that $A = \left(\begin{smallmatrix} a & -a-1 \\ 1 & -1 \end{smallmatrix}\right)$. Observe that

$$\left(\begin{smallmatrix} a & a-1 \\ 1 & -1 \end{smallmatrix}\right)(-\overline{\rho}) = a - \frac{1}{-\overline{\rho} - 1} = a - \overline{\rho}.$$

Thus we may either take $a = -1$, in which case $A = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and $A(-\overline{\rho}) = \rho$, or $a = 0$, in which case $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) = (TS)^2$ and $A(-\overline{\rho}) = -\overline{\rho}$.

– Suppose that $c = 1$, $d = 1$. Analogously to the above case, we find that this implies that $\tau = \rho$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ or $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) = ST$. In the former case we have $A\rho = -\overline{\rho}$, and in the latter we have $A\rho = \rho$.

This gives a complete classification of points in $\overline{\mathcal{D}}$ related by elements of $\mathrm{SL}(2,\mathbb{Z})$. In particular it details the stabilisers of each point. Propositions 3.18 and 3.19 summarise what we have learnt.

This also proves Theorem 3.17. □

Having found a fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathfrak{h}$, checking the following proposition is straightforward.

Proposition 3.20. $\mathrm{SL}(2,\mathbb{Z})$ *acts properly discontinuously on* $\mathfrak{h}$.

Proof. Let $\tau, \tau' \in \mathfrak{h}$. To show that the action of $\mathrm{SL}(2,\mathbb{Z})$ is properly discontinuous, we wish to find open neighbourhoods U of τ and V of τ' such that $\{A \in \mathrm{SL}(2,\mathbb{Z}) \mid$

$AU \cap V \neq \varnothing\}$ is finite. By Theorem 3.17 and Proposition 3.18, we may assume without loss of generality that τ, τ' lie in the set

$$\mathcal{D} \cup \{z \in \mathfrak{h} \mid |z| > 1, \operatorname{Re} z = \tfrac{1}{2}\} \cup \{z \in \mathfrak{h} \mid |z| = 1, 0 \leq \operatorname{Re} z \leq \tfrac{1}{2}\}.$$

If $\tau, \tau' \in \mathcal{D}$, then we may take $U = V = \mathcal{D}$. By Propositions 3.18 and 3.19, $A\mathcal{D} \cap \mathcal{D} \neq \varnothing$ if and only if $A = \pm I$. Thus $\{A \in \mathrm{SL}(2, \mathbb{Z}) \mid AU \cap V \neq \varnothing\}$ is finite.

More generally, we consider the set

$$R := \mathcal{D} \cup S\mathcal{D} \cup (TS)^2\mathcal{D} \cup TST\mathcal{D} \cup TS\mathcal{D} \cup T\mathcal{D}.$$

Computation shows this is the region given by the conditions

$$\{z \in \mathfrak{h} \mid -\tfrac{1}{2} < \operatorname{Re} z < \tfrac{3}{2}, |z+1| > 1, |z-2| > 1, |z - \tfrac{1}{3}| > \tfrac{1}{3}, |z - \tfrac{2}{3}| > \tfrac{1}{3}\},$$

or, perhaps more tractably, that depicted in Figure 3.2.

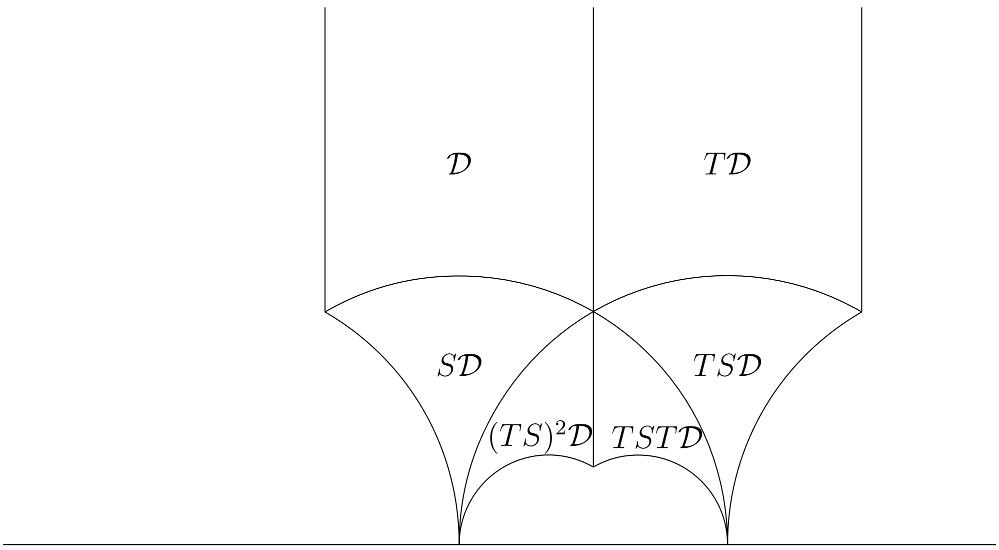


Figure 3.2: The domain R as a subset of the upper half plane $\mathfrak{h}$.

Clearly R is an open neighbourhood of τ and τ'. It is thus enough to show that the set $\{A \in \mathrm{SL}(2, \mathbb{Z}) \mid AR \cap R \neq \varnothing\}$ is finite. Suppose then that we have $A \in \mathrm{SL}(2, \mathbb{Z})$ such that $AR \cap R \neq \varnothing$. Then there exists $\sigma \in R$ such that $A\sigma \in R$. By the construction of R, we may find

$$B, C \in \{I, S, T, TS, TST, (TS)^2\}$$

such that $B^{-1}\sigma$ and $C^{-1}A\sigma$ are in $\overline{\mathcal{D}}$. This implies that $D = C^{-1}AB$ is an element of $\mathrm{SL}(2, \mathbb{Z})$ such that $B^{-1}\sigma$ and $D(B^{-1}\sigma)$ lie in $\overline{\mathcal{D}}$. Propositions 3.18 and

3.19 show that only finitely many such elements exist. Thus $A = CDB^{-1}$, where only finitely many possibilities for B, C and D exist. Therefore there are only finitely many A such that $AR \cap R \neq \varnothing$. This shows that the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathfrak{h}$ is properly discontinuous. □

The sets U and V in the above proof were constructed rather crudely. Using the argument of Lemma 3.14, it is possible to construct U and V such that $\{A \in \mathrm{SL}(2, \mathbb{Z}) \mid AU \cap V \neq \varnothing\} = \{A \in \mathrm{SL}(2, \mathbb{Z}) \mid A\tau = \tau'\}$. For such sets U and V the set $\{A \in \mathrm{SL}(2, \mathbb{Z}) \mid AU \cap V \neq \varnothing\}$ is empty when τ and τ' are not equivalent, and of order equal to the order of $\mathrm{Stab}(\tau)$ when τ and τ' are equivalent.

Having established Proposition 3.20, Theorem 3.11 allows us to view $\mathcal{M}$ as a complex manifold, and in such a way that the quotient map $\mathfrak{h} \to \mathcal{M}$ is holomorphic. This establishes the theorem:

Theorem 3.21. *The space $\mathcal{M}$ is a coarse moduli space for elliptic curves.*

Another method of proving this is via consideration of the so-called j-invariant. This gives an explicit biholomorphism between $\mathcal{M}$ and $\mathbb{C}$. Details can be found in Clemens [4, §3.12] or Silverman [29, §1.4]

3.5 Automorphisms of Elliptic Curves

We have seen that automorphisms obstruct the construction of a fine moduli space. This suggests that it is useful to know the automorphisms of lattices and elliptic curves. We can quite easily glean this information from the coarse moduli space.

Proposition 3.22. *Let $(E; O)$ be an elliptic curve, and let $\tau \in \mathfrak{h}$ be such that $(E; O) \cong (\mathbb{C}/\varphi_\tau(\mathbb{Z}^2); 0)$. Then there is a canonical isomorphism of groups*

$$\mathrm{Aut}(E; O) \cong \mathrm{Stab}_{\mathrm{SL}(2,\mathbb{Z})}(\tau).$$

Proof. As the categories of elliptic curves and lattices are equivalent, the group of automorphisms of the elliptic curve $(E; O)$ is canonically isomorphic to the group of automorphisms of the corresponding lattice $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau)$. This is then an immediate consequence of Theorem 3.10. □

From Proposition 3.19, we can thus conclude from this that up to isomorphism the only elliptic curves with automorphisms other than the trivial automorphism and the negation automorphism are those corresponding to the points i and ρ. As lattices, we may visualise these as in Figure 3.3.

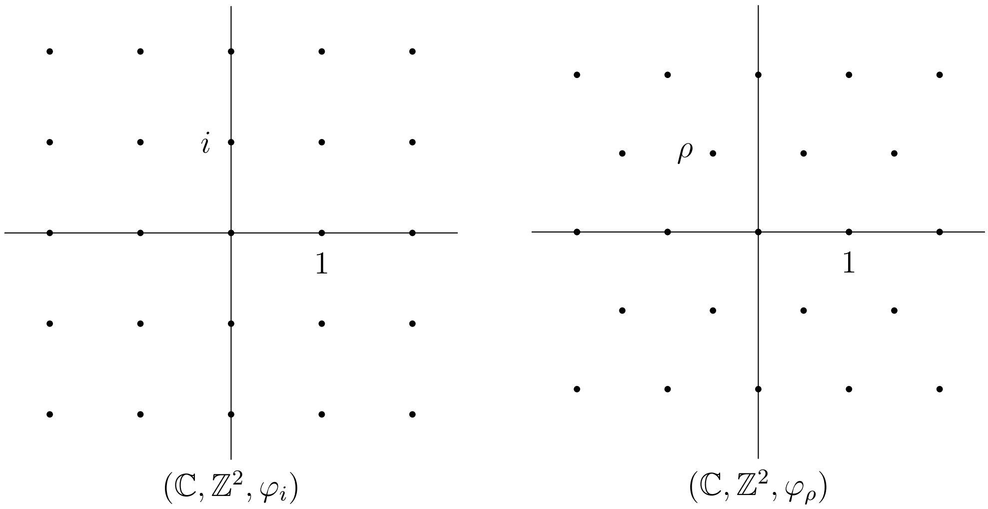


Figure 3.3: The lattices with extra automorphisms.

Geometrically the automorphisms of these lattices correspond to rotations; the groups are generated by a rotation by $\pi/2$ and $\pi/3$, respectively. Indeed, the fact that there are no other elliptic curves with automorphisms is equivalent to the fact that these are the only lattices in $\mathbb{C}$ generated by 1 and $\tau \in \overline{\mathcal{D}}$ with rotational symmetry other than the reflection given by rotation by π.

3.6 Curves of Higher Genus

The techniques of this chapter extend more generally to other situations. One such situation is the construction of a (coarse) moduli space for compact Riemann surfaces of genus greater than 1. This brief discussion follows that of Hain [15, Lect. 2]. Further detail can be found in Harris and Morrison [16].

We first take the following perspective on the definition of lattices. Recall that the universal cover of any genus 1 curve is biholomorphic to $\mathbb{C}$, and that the automorphism group is the complex affine group

$$\mathrm{Aut}\mathbb{C} = \{z \mapsto \alpha z + \beta \mid \alpha, \beta \in \mathbb{C}, \alpha \neq 0\}.$$

Let $(E; O)$ be an elliptic curve. Considering an element of $\tau \in \mathbb{C}$ as an element $z \mapsto z + \tau$ of $\mathrm{Aut}\mathbb{C}$, and noting that the first homotopy and homology groups

$\pi_1(E, O)$ and $H_1(E; \mathbb{Z})$ of an elliptic curve are isomorphic, in the construction of the period lattice we have seen that we can construct an injective group homomorphism

$$\varphi : \pi_1(E, O) \longrightarrow \mathrm{Aut}\mathbb{C}$$

such that $\mathrm{Im}\,\varphi$ acts freely on $\mathbb{C}$ and $\mathrm{Im}\,\varphi\backslash\mathbb{C}$ is a compact Riemann surface of genus 1. This homomorphism, as the data for a lattice, is unique up to multiplication by a complex scalar. Noting that conjugation of φ by an element of $\mathrm{Aut}\mathbb{C}$ is equivalent to multiplication by a complex scalar, we thus see that we naturally associate to the elliptic curve $(E; O)$ a conjugacy class of injective maps φ from its fundamental group to the group of automorphisms of its universal cover.

For Riemann surfaces of higher genus, we can make an analogous construction in the following way. The uniformisation theorem for compact Riemann surfaces states that the universal cover of any compact Riemann surface of genus greater than 1 is $\mathfrak{h}$, and it can be shown that the group of holomorphic automorphisms of $\mathfrak{h}$ is naturally isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Let X_g be a compact Riemann surface of genus $g \geq 2$, and x_0 be a point in X_g. Then we define the gth Teichmüller space to be

$$\chi_g := \left\{ \begin{array}{c} \mathrm{PSL}(2, \mathbb{R})\text{-conjugacy classes of injective representations} \\ \varphi : \pi_1(X_g, x_0) \to \mathrm{PSL}(2, \mathbb{R}) \text{ such that } \mathrm{Im}\,\varphi \text{ acts freely on } \mathfrak{h} \\ \text{and } \mathrm{Im}\,\varphi\backslash\mathfrak{h} \text{ is a compact Riemann surface of genus } g \end{array} \right\}.$$

Just as we associate to each point in the upper half plane $\mathfrak{h}$ a based lattice, we may associate to each point $[\varphi]$ of χ_g the Riemann surface $\mathrm{Im}\,\varphi\backslash\mathfrak{h}$, which is well-defined up to isomorphism.

We also wish for a group that plays a role analogous to $\mathrm{SL}(2, \mathbb{Z})$. This group should tell us when the compact Riemann surfaces associate to two different points of χ_g are isomorphic. Such information is given by the group

$$\Gamma_g := \left\{ \begin{array}{c} \text{connected components of the group of orientation} \\ \text{preserving diffeomorphisms of a surface of genus } g \end{array} \right\}.$$

This group depends only on the genus as all genus g surfaces are diffeomorphic.

It can be shown that for all $g \geq 2$ the Teichmüller space χ_g is contractible and the mapping class group Γ_g acts properly discontinuously. We have shown that this realises $\mathcal{M}_g$ at least as a topological space. Since χ_g is not necessarily a Riemann surface—indeed, it can be shown that it is naturally a simply-connected open subset of $\mathbb{C}^{3g-3}$ and hence of dimension $3g - 3$—the quotient $\Gamma_g\backslash\chi_g$ is not necessarily a complex manifold. It is can be realised, however, as an analytic variety—a complex manifold with some singularities. This space is a coarse moduli space for compact Riemann surfaces of genus g. It is also true that the stabiliser of $[\varphi] \in \chi_g$ is naturally isomorphic to the group of automorphisms of the Riemann surface $\operatorname{Im}\varphi\backslash\mathfrak{h}$.

While the results of this chapter have given a method for constructing coarse moduli spaces, however, they do not give an indication of whether the structure of a fine moduli space descends. The next chapter addresses this issue.

Chapter 4

Level Structures

So far our discussion has focussed on taking quotients of the moduli space of based lattices: we have ignored the universal family above this space. We now examine the conditions under which this universal family descends to a universal family on the quotient space. Although this cannot give a construction of a moduli space for elliptic curves, it can give a moduli space for elliptic curves with only a slight rigidification. We call these rigidifications level structures.

4.1 Identifying Oriented Based Lattices

We have already defined an action of $\mathrm{SL}(2,\mathbb{Z})$ on the moduli space $\mathfrak{h}$ of oriented based lattices. We now wish to extend this action to an action of $\mathrm{SL}(2,\mathbb{Z})$ on the universal family of oriented based lattices.

From our discussions on families of based lattices and also on their relationship with families of oriented based lattices, it can be seen that the universal family of oriented based lattices is given by the triple $(\mathbb{C}\times\mathfrak{h},\mathfrak{h},\Omega)$, where $\mathbb{C}\times\mathfrak{h}$ is the trivial line bundle over $\mathfrak{h}$ and Ω is defined by

$$\begin{aligned} \Omega : \mathbb{Z}^2\times\mathfrak{h} &\longrightarrow \mathbb{C}\times\mathfrak{h}; \\ \big((m,n),\tau\big) &\longmapsto (m\tau+n,\tau). \end{aligned}$$

This gives the commutative triangle:

$$\begin{array}{ccc} \mathbb{Z}^2\times\mathfrak{h} & \xrightarrow{\ \Omega\ } & \mathbb{C}\times\mathfrak{h} \\ & {\scriptstyle \mathrm{proj}_2}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}_2} & \\ & \mathfrak{h}. & \end{array}$$

Also recall that given $\tau \in \mathfrak{h}$, the fibre over this family is the based lattice $(\mathbb{C}, \varphi_\tau)$, and that by Theorem 3.10, given $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ the fibres over τ and $\tau' = A\tau$ are isomorphic by

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{\varphi_\tau} & \mathbb{C} \\ {\scriptstyle f_\Lambda^{A,\tau}}\big\downarrow & & \big\downarrow{\scriptstyle f_V^A} \\ \mathbb{Z}^2 & \xrightarrow{\varphi_{\tau'}} & \mathbb{C} \end{array}$$

where

$$\begin{aligned} f_V^{A,\tau} : \ \mathbb{C} &\longrightarrow \mathbb{C}; \\ z &\longmapsto \tfrac{z}{c\tau+d}, \\ f_\Lambda^A : \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2; \\ (m,n) &\longmapsto (m,n)A^{-1} = (dm - cn, -bm + an). \end{aligned}$$

This motivates the (left-)actions

$$\begin{aligned} A : \mathbb{Z}^2 \times \mathfrak{h} &\longrightarrow \mathbb{Z}^2 \times \mathfrak{h}; \\ \big((m,n),\tau\big) &\longmapsto \Big(f_\Lambda^A(m,n), A\tau\Big) \\ &\qquad = \big((dm - cn, -bm + an), \tfrac{a\tau+b}{c\tau+d}\big) \end{aligned}$$

and

$$\begin{aligned} A : \mathbb{C} \times \mathfrak{h} &\longrightarrow \mathbb{C} \times \mathfrak{h}; \\ \big(z,\tau\big) &\longmapsto \Big(f_V^{A,\tau}(z), A\tau\Big) = \big(\tfrac{z}{c\tau+d}, \tfrac{a\tau+b}{c\tau+d}\big), \end{aligned}$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2 \times \mathfrak{h}$ and $\mathbb{C} \times \mathfrak{h}$. It is evident that these functions are holomorphic, and it is easily checked that these indeed define left actions.

Interpreted geometrically, these actions change the coordinates (m,n) of a point $m\tau + n \in \mathbb{C}$ to the coordinates $(dm - cn, -bm + an)$ of the same point with respect to the basis $(a\tau + b, c\tau + d)$, before scaling such that the second element of this basis is equal to 1. In this way a point of a lattice $\varphi_\tau(\mathbb{Z}^2)$ becomes a point of the lattice $\varphi_{\tau'}(\mathbb{Z}^2)$.

We would like the quotient of this family by this $\mathrm{SL}(2,\mathbb{Z})$-action to be a family of lattices over the quotient $\mathcal{M} = \mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ of the base space, and in such a way that the fibre over $\tau \in \mathfrak{h}$ is naturally isomorphic to the fibre over its image in

$\mathcal{M}$. Suppose, however, a fibre $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau)$ has a nontrivial automorphism. Then, again citing Theorem 3.10, there exists some nonidentity $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $A\tau = \tau$ and A acts on the fibre $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau)$ by this nontrivial automorphism. If this is so, upon taking the quotient by $\mathrm{SL}(2,\mathbb{Z})$, distinct points of this fibre are identified, and so the fibre is not 'preserved' in the quotient. We hence see again that nontrivial automorphisms of our objects interfere.

This can be avoided if the action is free on the base space. We shall see, in fact, that in this case the universal family on $\mathfrak{h}$ always descends to a family of lattices on the quotient of $\mathfrak{h}$.

4.2 Descent of Bundles

As families of lattices are constructed from bundles, we shall discuss first the theory involved in descending families in this more general context. The main theorem of this section is that bundles on a quotient of a given space can be considered as special types of bundles on the given space.

Consider the action of $\mathrm{SL}(2,\mathbb{Z})$ on the universal family of oriented based lattices. Observe that, given $A \in \mathrm{SL}(2,\mathbb{Z})$ and an element $((m,n),\tau)$ of $\mathbb{Z}^2 \times \mathfrak{h}$, it matters not whether we have A act on $((m,n),\tau)$ and then project to the base space $\mathfrak{h}$, or project to the base space and then have A act on the projection—in both cases we get the element $A\tau \in \mathfrak{h}$. Similarly the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{C} \times \mathfrak{h}$ commutes with its projection onto $\mathfrak{h}$. Furthermore, given any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and any $((m,n),\tau) \in \mathbb{Z}^2 \times \mathfrak{h}$, we have

$$\begin{aligned}\Omega(A \cdot ((m,n),\tau)) &= \Omega((m,n)A^{-1}, A\cdot\tau) \\ &= \left((m,n)A^{-1}\left(\tfrac{1}{c\tau+d}A\left(\begin{smallmatrix}\tau \\ 1\end{smallmatrix}\right)\right), A\cdot\tau\right) \\ &= \left(\tfrac{m\tau+n}{c\tau+d}, \tfrac{a\tau+b}{c\tau+d}\right)\end{aligned}$$

and

$$A \cdot \Omega(((m,n),\tau)) = A \cdot (m\tau + n, \tau) = \left(\tfrac{m\tau+n}{c\tau+d}, \tfrac{a\tau+b}{c\tau+d}\right).$$

Thus for all such A we $\Omega(A \cdot ((m,n),\tau))) = A \cdot \Omega((m,n),\tau))$. In all such cases we say that the maps are $\mathrm{SL}(2,\mathbb{Z})$-equivariant.

More generally, a map $f : X \to Y$ between two spaces X and Y each equipped with a Γ-action is called Γ-*equivariant* if for all $\gamma \in \Gamma$ the square

$$\begin{array}{ccc} X & \xrightarrow{\gamma\cdot} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\gamma\cdot} & Y \end{array}$$

commutes. In this case, we may then define a map

$$f_\Gamma : \Gamma\backslash X \longrightarrow \Gamma\backslash Y$$

sending the Γ-orbit of a point $x \in X$ to the Γ-orbit of $f(x) \in Y$. This is well-defined as the Γ-equivariance of f implies that x and y lie in the same Γ-orbit only if $f(x)$ and $f(y)$ lie in the same Γ-orbit. This motivates the following definitions.

Definitions 4.1. Let Γ be a group acting on a complex manifold B. We then call a bundle (E, B, F, π) over B a Γ-*bundle over* B[1] if we have in addition a Γ-action on E, and furthermore the projection map π is Γ-equivariant.

Given two Γ-bundles (E_1, B, F_1, π_1) and (E_2, B, F_2, π_2) over B and a bundle map Φ between them, we call Φ a *morphism of* Γ-*bundles* if Φ is Γ-equivariant. Observe that it is implicit here that the action of Γ on B is the same for both bundles—the term 'Γ-bundle over B' assumes the Γ-action on B is already determined.

These objects and morphisms form a category; we denote this category Γ-$\mathbf{Bund}_B$. More generally we shall denote the category of bundles over a complex manifold B as $\mathbf{Bund}_B$. We are, as usual, especially interested in the case when the actions of Γ are free and properly discontinuous. The main theorem of this section is the following.

Theorem 4.2. *Let Γ be a group acting freely and properly discontinuously on a complex manifold B. Then the categories* Γ-$\mathbf{Bund}_B$ *and* $\mathbf{Bund}_{\Gamma\backslash B}$ *are equivalent.*

Suppose we have a Γ-bundle (E, B, F, π) over B. A corollary of the following lemma is that when the action of Γ on B is free and properly discontinuous, then it is immediate that the action of Γ on E is free and properly discontinuous.

[1] This convention differs from much of the literature—the term Γ-bundle is often used to refer to a fibre bundle with structure group Γ.

Lemma 4.3. *Let Γ be a group acting on topological spaces X and Y, and let $f : X \to Y$ be a continuous Γ-equivariant map. Then* (i) *if Γ acts freely on Y, then Γ acts freely on X, and* (ii) *If Γ acts properly discontinuously on Y, then Γ acts properly discontinuously on X.*

Proof. (i) Suppose Γ acts freely on Y. Suppose also that we have $\gamma \in \Gamma$ and $x \in X$ such that $\gamma \cdot x = x$. Then $\gamma \cdot f(x) = f(x)$, and so γ is the identity of Γ. This proves that Γ acts freely on X.

(ii) Suppose Γ acts properly discontinuously on Y. Let $x_1, x_2 \in X$. We wish to find open neighbourhoods U_1 and U_2 of x_1 and x_2 respectively such that the set

$$\{\gamma \in \Gamma \mid \gamma U_1 \cap U_2 \neq \varnothing\}$$

is finite. Since Γ acts properly discontinuously on Y, there exist open neighbourhoods V_1 and V_2 of $f(x_1)$ and $f(x_2)$ such that $\{\gamma \in \Gamma \mid \gamma V_1 \cap V_2 \neq \varnothing\}$ is finite. Set $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Then, by the Γ-equivariance of f, for all $\gamma \in \Gamma$ such that $\gamma U_1 \cap U_2 \neq \varnothing$ we have $\gamma f(U_1) \cap f(U_2) = \gamma V_1 \cap V_2 \neq \varnothing$. Thus $\{\gamma \in \Gamma \mid \gamma U_1 \cap U_2 \neq \varnothing\}$ is contained in $\{\gamma \in \Gamma \mid \gamma V_1 \cap V_2 \neq \varnothing\}$, and hence finite.

□

In order to prove the main theorem, we will construct functors between the two categories, and then show that these functors are adjoints. The part of most interest to us is the descent.

Proposition 4.4. *Let Γ be a group acting freely and properly discontinuously on a complex manifold B, and let (E, B, F, π) be a Γ-bundle. Then $(\Gamma\backslash E, \Gamma\backslash B, F, \pi_\Gamma)$ is a fibre bundle over $\Gamma\backslash B$.*

Furthermore, if $\Phi : E_1 \to E_2$ is a morphism of Γ-bundles (E_1, B, F_1, π_1) and (E_2, B, F_2, π_2) over B, then $\Phi_\Gamma : \Gamma\backslash E_1 \to \Gamma\backslash E_2$ is a morphism of the bundles $(\Gamma\backslash E_1, \Gamma\backslash B, F_1, (\pi_1)_\Gamma)$ and $(\Gamma\backslash E_2, \Gamma\backslash B, F_2, (\pi_2)_\Gamma)$ over $\Gamma\backslash B$.

Proof. We first show that $(\Gamma\backslash E, \Gamma\backslash B, F, \pi_\Gamma)$ is a fibre bundle. By Lemma 4.3 the actions of Γ on E and B are free and properly discontinuous. Lemma 0.15 thus shows that $\Gamma\backslash E$ and $\Gamma\backslash B$ are complex manifolds, and in fact that the quotient maps $q_E : E \to \Gamma\backslash E$ and $q_B : B \to \Gamma\backslash B$ are locally biholomorphic covering maps. This is the key fact in what follows.

The first property to check is that π_Γ is a holomorphic surjection. By definition, $\pi : E \to B$ is a holomorphic surjection. It is immediate from this that π_Γ is a surjection. Consider the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{q_E} & \Gamma\backslash E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_\Gamma} \\ B & \xrightarrow{q_B} & \Gamma\backslash B. \end{array}$$

Since q_E is a locally biholomorphic covering map, around any point $e \in \Gamma\backslash E$, we can find an open neighbourhood U in $\Gamma\backslash E$ and an open set $\widetilde{U}$ in E such that $q|_{\widetilde{U}}$ is a biholomorphism. We then have

$$\pi_\Gamma|_U = q_B \circ \pi \circ (q_E|_{\widetilde{U}})^{-1}.$$

Since the functions on the right hand side are all holomorphic, π_Γ must be holomorphic about e. Since e was arbitrary, this shows that π_Γ is holomorphic.

A similar technique can be applied to find a local trivialisation. Let $x \in \Gamma\backslash B$. We wish to find an open set U of $\Gamma\backslash B$ containing x such that $\pi_\Gamma^{-1}(U)$ is biholomorphic to $U \times F$. Choose U such that there exists an open set $\widetilde{U}$ of B with $q_B|_{\widetilde{U}} : \widetilde{U} \to U$ a biholomorphism. By replacing U by a subset if necessary, we may assume that $\widetilde{U}$ is a trivialising neighbourhood for the fibre bundle (E, B, F, π). Furthermore, since no two points of $\widetilde{U}$ lie in the same Γ-orbit and π is Γ-equivariant, no two points of $\pi^{-1}(\widetilde{U})$ lie in the same Γ-orbit. This means the map $q_E|_{\pi^{-1}(\widetilde{U})}$ is injective, and hence a biholomorphism onto its image. We thus have

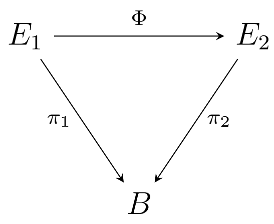
$$\pi_\Gamma^{-1}(U) = q_E(\pi^{-1}(\widetilde{U})) \cong \pi^{-1}(\widetilde{U}) \cong F \times \widetilde{U} \cong F \times U.$$

This proves that $(\Gamma\backslash E, \Gamma\backslash B, F, \pi_\Gamma)$ is a fibre bundle.

To see that morphisms descend, observe first that Φ_Γ is holomorphic by the same lifting argument that shows that the projection π_Γ is holomorphic. It thus suffices to check that the diagram

$$\begin{array}{ccc} \Gamma\backslash E_1 & \xrightarrow{\Phi_\Gamma} & \Gamma\backslash E_2 \\ & {\scriptstyle(\pi_1)_\Gamma}\searrow \quad \swarrow{\scriptstyle(\pi_2)_\Gamma} & \\ & \Gamma\backslash B & \end{array}$$

commutes. But this is clear from fact that



commutes and the definitions of the quotients and their maps. $\square$

This shows we can define a quotient functor $Q : \Gamma\text{-}\mathbf{Bund}_B \to \mathbf{Bund}_{\Gamma\backslash B}$, given by simply taking quotients with respect to the Γ-action. To move in the reverse direction we use pullbacks. Suppose we have sets X, Y and Y', maps $f : Y' \to Y$ and $\pi : X \to Y$, and suppose a group Γ acts on Y'. Then the pullback

$$X \times_Y Y' = \{(x, y) \in X \times Y' \mid f(x) = \pi(y)\}$$

comes equipped with the Γ-action defined by

$$\gamma : (x, y) \longmapsto (x, \gamma y),$$

where $\gamma \in \Gamma$.

Proposition 4.5. *Let Γ be a group that acts freely and properly discontinuously on a complex manifold B, and let $q : B \to \Gamma\backslash B$ be the quotient map. Let $(E, \Gamma\backslash B, F, \pi)$ be a fibre bundle over $\Gamma\backslash B$. Then the pullback bundle $(E \times_{\Gamma\backslash B} B, B, F, \widehat{\pi})$ is a Γ-bundle over B.*

Furthermore, if $\Phi : E_1 \to E_2$ is a morphism of bundles $(E_1, \Gamma\backslash B, F_1, \pi_1)$ and $(E_2, \Gamma\backslash B, F_2, \pi_2)$ over $\Gamma\backslash B$, then $\widehat{\Phi} : E_1 \times_{\Gamma\backslash B} B \to E_2 \times_{\Gamma\backslash B} B$ is a morphism of the Γ-bundles $(E_1 \times_{\Gamma\backslash B} B, B, F_1, \widehat{\pi}_1)$ and $(E_2 \times_{\Gamma\backslash B} B, B, F_2, \widehat{\pi}_2)$ over B.

Proof. Since we know the pullback of a bundle is again a bundle, we need only show that the pullback $\widehat{\pi} : E \times_{\Gamma\backslash B} B \to B$ of the projection map is Γ-equivariant. Observe that for any $\gamma \in \Gamma$ and $(e, b) \in E \times_{\Gamma\backslash B} B$, we have

$$\widehat{\pi}(\gamma(e, b)) = \widehat{\pi}(e, \gamma b) = \gamma b = \gamma\widehat{\pi}(e, b).$$

This shows we have a Γ-bundle.

Again we know that the pullback of a bundle map is again a bundle map, so it suffices to check that the pullback is Γ-equivariant. This is true: for all $\gamma \in \Gamma$ and $(e, b) \in E_1 \times_{\Gamma\backslash B} B$, we have

$$\widehat{\Phi}(\gamma(e, b)) = \widehat{\Phi}(e, \gamma b) = (\Phi(e), \gamma b) = \gamma(\Phi(e), b) = \gamma\widehat{\Phi}(e, b).$$

This shows the pullback of a morphism of bundles over $\Gamma\backslash B$ is a morphism of Γ-bundles over B, as required. □

We may thus define a functor $P : \mathbf{Bund}_{\Gamma\backslash B} \to \Gamma\text{-}\mathbf{Bund}_B$ that sends bundles and morphisms to their pullbacks along the quotient map $q : B \to \Gamma\backslash B$. Having defined our functors, we now prove the main theorem.

Proof of Theorem 4.2. To prove the claimed category equivalence it suffices to exhibit natural isomorphisms $\alpha : \mathbf{id}_{\mathbf{Bund}_{\Gamma\backslash B}} \dot{\to} Q \circ P$ and $\beta : \mathbf{id}_{\Gamma\text{-}\mathbf{Bund}_B} \dot{\to} P \circ Q$.

Define a natural transformation $\alpha : \mathbf{id}_{\mathbf{Bund}_{\Gamma\backslash B}} \dot{\to} Q \circ P$ by assigning the bundle map

$$\begin{aligned} \alpha_E : E &\longrightarrow \Gamma\backslash(E \times_{\Gamma\backslash B} B); \\ e &\longmapsto [e, b] \end{aligned}$$

to every bundle $(E, \Gamma\backslash B, F\pi)$ over $\Gamma\backslash B$, where b is any point in B such that $q(b) = \pi(e)$ and $[e, b]$ indicates the Γ-orbit of the point $(e, b) \in \widetilde{X} \times_X E$. Also define a natural transformation $\beta : \mathbf{id}_{\Gamma\text{-}\mathbf{Bund}_B} \dot{\to} P \circ Q$ by assigning the Γ-bundle morphism

$$\begin{aligned} \alpha_E : E &\longrightarrow E/\Gamma \times_{\Gamma\backslash B} B; \\ e &\longmapsto ([e], \pi(e)), \end{aligned}$$

where $[e]$ is the Γ-orbit of e, to every Γ-bundle (E, B, F, π) over B. It is easy to check that these two natural transformations are well-defined, and are in fact natural isomorphisms. □

Observe that given a bundle, the data required for that bundle to be a group bundle can be expressed entirely in terms of commutative diagrams in the category of bundles. Since the categories $\Gamma\text{-}\mathbf{Bund}_B$ and $\mathbf{Bund}_{\Gamma\backslash B}$ are equivalent, this shows that group objects in one correspond to group objects in the other. Note, however, that as all maps in $\Gamma\text{-}\mathbf{Bund}_B$ are Γ-equivariant, the maps defining a group object in this category must also be Γ-equivariant. Similarly, as the data for a vector bundle can also be specified diagrammatically, the vector space objects in one category correspond to vector space objects in the other. This motivates the following definition:

Definitions 4.6. Let Γ be a group acting on complex manifolds B, $\mathcal{V}$ and $\mathcal{L}$. We say a family of lattices $(\mathcal{V}, \mathcal{L}, B, \Phi)$ is a Γ-*family of lattices* if $\mathcal{V}$ and $\mathcal{L}$ are Γ-bundles over B, the maps defining the group structure on $\mathcal{L}$ and vector space structure on $\mathcal{V}$ are Γ-equivariant, and Φ is Γ-equivariant.

The above discussion shows that Γ-families of lattices over B correspond to families of lattices over $\Gamma\backslash B$. Noting that the universal family of oriented based lattices is an $\mathrm{SL}(2,\mathbb{Z})$-family of lattices, this shows that if a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ acts freely on $\mathfrak{h}$, then this universal family descends to a family of lattices on the quotient $\Gamma\backslash\mathfrak{h}$. This principle guides our search for a level structure.

4.3 Level Structures

If our aim is only to remove automorphisms of a lattice, choosing a basis is a rather inefficient. We have seen that a lattice has at most six automorphisms, and yet there are infinitely many bases we may choose for any given lattice. Level structures give an alternate, less harsh, way of rigidifying lattices.

Once we have chosen a level structure, we might then investigate when distinct based lattices are isomorphic as lattices with level structure. Since isomorphic lattices with level structure must at least be isomorphic as lattices, we may describe these identifications of based lattices via a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$. As we have said, we will aim to choose level structures that result in the group Γ acting freely on $\mathfrak{h}$, so that the universal family of oriented based lattices descends to a family of lattices on the quotient $\Gamma\backslash\mathfrak{h}$. We will see that such a family is in fact the universal family of lattices with the chosen level structure. This, however, will be left to the next section—we first discuss some possible level structures.

N-Pointed Lattices

One method of constructing level structures for lattices is specifying an 'N-point'. This will be our main example of a level structure. Roughly speaking, an N-point is a point of order N in the vector space modulo the lattice. We make the following definitions.

Definitions 4.7. Let (V, Λ, φ) be a lattice and N be a positive integer. We call a subset P of V an *N-division point*, or N-point, of the lattice (V, Λ, φ) if it is equal to the coset $[p] := p + \varphi(\Lambda)$ of p for some $p \in V$ and N is the least positive integer such that Np lies in $\varphi(\Lambda)$. We call the data (V, Λ, φ, P) an *N-pointed lattice.*

A *morphism of N-pointed lattices* $(V_1, \Lambda_1, \varphi_1, P_1)$ and $(V_2, \Lambda_2, \varphi_2, P_2)$ is a map of lattices (f_Λ, f_V) such that $f_V(P_1) = P_2$.

Example 4.8. Let (V, φ) be a based lattice. We may assign an N-division point to this lattice by taking the coset $[\frac{1}{N}\varphi(0,1)]$ of the point 'one Nth of the second basis vector'. We take the convention that any based lattice is by default viewed as an N-pointed lattice in this way, and call this coset the *N-division point associated to the based lattice* (V, φ). Conversely, given an N-pointed lattice (V, Λ, φ, P), a basis λ_1, λ_2 for Λ such that $\frac{1}{N}\varphi(\lambda_2) \in P$ is said to *extend the N-division point.*

In particular, this convention assigns the point $[\frac{1}{N}]$ to the based lattice $(\mathbb{C}, \varphi_\tau)$, and we thus view this based lattice as the N-pointed lattice $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$.

Just as a based lattice corresponds to an elliptic curve with a basis for its first homology group $H_1(E; \mathbb{Z})$ with coefficients in $\mathbb{Z}$, an N-pointed lattice may be interpreted as an elliptic curve together with a point of order N in the homology group $H_1(E; \mathbb{Z}/N\mathbb{Z})$ with coefficients in $\mathbb{Z}/N\mathbb{Z}$.

It can be shown that a given lattice has exactly $N^2 \prod_{p|N}(1 - 1/p^2)$ N-division points; this is simply a computation of the elements of exact order N in the group $(\mathbb{Z}/N\mathbb{Z})^2$. In particular, there are only eight ways to turn a lattice into a 3-pointed lattice—this contrasts starkly with the infinitely many bases that can be assigned to the lattice.

The following proposition gives a better idea of how N-pointed lattices transform. As for based lattices, all morphisms are isomorphisms.

Proposition 4.9. *Any morphism of N-pointed lattices is an isomorphism of N-pointed lattices.*

Proof. Let (f_Λ, f_V) be a morphism of N-pointed lattices $(V_1, \Lambda_1, \varphi_1, [p_1])$ and $(V_2, \Lambda_2, \varphi_2, [p_2])$. It suffices to show that both f_Λ and f_V are isomorphisms.

Since $f_V([p_1]) = [p_2]$, f_V is a nonzero map of 1-dimensional complex vector spaces, it is an isomorphism of vector spaces. Furthermore, as $f_V \circ \varphi_1 = \varphi_2 \circ f_\Lambda$, this shows that f_Λ is injective. It remains to show that f_Λ is surjective. Again as $f_V([p_1]) = [p_2]$, we have the equality of cosets

$$f_V(p_1) + f_V(\varphi_1(\Lambda_1)) = p_2 + \varphi_2(\Lambda_2),$$

and hence the equality of subgroups

$$f_V(\varphi_1(\Lambda_1)) = \varphi_2(\Lambda_2)$$

of V_2. The left hand group is then equal to $\varphi_2(f_\Lambda(\Lambda_1))$, and so the injectivity of φ_2 then gives the equality $f_\Lambda(\Lambda_1) = \Lambda_2$. This proves the surjectivity of f_Λ, and hence the proposition. □

The previous example gave a way of constructing an N-pointed lattice from a based lattice. Up to isomorphism, all N-pointed lattices arise in this way.

Proposition 4.10. *Let* $(V, \Lambda, \varphi, [p])$ *be an* N*-pointed lattice. Then there exists* $\tau \in \mathfrak{h}$ *such that* $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$ *is isomorphic to* $(V, \Lambda, \varphi, [p])$ *as an* N*-pointed lattice.*

Proof. Since $[p]$ is an N-division point for (V, Λ, φ), Np lies in $\varphi(\Lambda)$. As Λ is a rank two free abelian group, we may choose a basis λ_1, λ_2 for Λ such that $\lambda_2 = \varphi^{-1}(Np)$. Replacing λ_1 by $-\lambda_1$ if necessary, we may also assume that $0 < \arg(\varphi(\lambda_1)/\varphi(\lambda_2)) < \pi$. There then exists a unique complex number $\tau \in \mathfrak{h}$ such that $\varphi(\lambda_1) = \tau\varphi(\lambda_2)$. We shall show that for this τ the N-pointed lattices $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$ and $(V, \Lambda, \varphi, [p])$ are isomorphic.

Let $f_\Lambda : \Lambda \to \mathbb{Z}^2$ be the group isomorphism mapping λ_1 to $(1, 0)$ and λ_2 to $(0, 1)$, and let $f_V : V \to \mathbb{C}$ be the unique $\mathbb{C}$-linear map taking p to $\frac{1}{N}$. This defines a morphism of lattices as for all $m, n \in \mathbb{Z}$ we have

$$\begin{aligned} f_V(\varphi(m\lambda_1 + n\lambda_2)) &= f_V(m\varphi(\lambda_1) + n\varphi(\lambda_2)) \\ &= f_V((m\tau + n)Nv) \\ &= m\tau + n \\ &= \varphi_\tau(m, n) \\ &= \varphi_\tau(f_\Lambda(m\lambda_1 + n\lambda_2)). \end{aligned}$$

Furthermore, we thus have the equalities of sets

$$\begin{aligned} f_V([p]) = f_V(p + \varphi(\Lambda)) &= f_V(p) + f_V(\varphi(\Lambda)) \\ &= \tfrac{1}{N} + \varphi_\tau(f_\Lambda(\Lambda)) = \tfrac{1}{N} + \varphi_\tau(\mathbb{Z}^2) = [\tfrac{1}{N}]. \end{aligned}$$

This shows that (f_V, f_Λ) is a morphism of N-pointed lattices, and hence an isomorphism of N-pointed lattices. □

Suppose that $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$ and $(\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'}, [\frac{1}{N}])$ are isomorphic N-pointed lattices. A fortiori they are isomorphic as lattices, so by Theorem 3.10 there exists a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ such that the isomorphism is given by $(f_V^{A,\tau}, f_\Lambda^A)$. The matrix A, however, is subject to the extra condition that

$$f_V^{A,\tau}\Big(\tfrac{1}{N} + \varphi_\tau(\mathbb{Z}^2)\Big) = \tfrac{1}{N} + \varphi_{\tau'}(\mathbb{Z}^2).$$

Composing with $(f_V^{A,\tau})^{-1}$ and recalling that this maps z in $\mathbb{C}$ to $(c\tau + d)z$, this implies that

$$\tfrac{1}{N} + \varphi_\tau(\mathbb{Z}^2) = (f_V^{A,\tau})^{-1}\Big(\tfrac{1}{N} + \varphi_{\tau'}(\mathbb{Z}^2)\Big) = (c\tau + d)\tfrac{1}{N} + \varphi_\tau(\mathbb{Z}^2).$$

Subtracting $\frac{1}{N}$, this shows that

$$\tfrac{c}{N}\tau + \tfrac{d-1}{N} = (c\tau + d)\tfrac{1}{N} - \tfrac{1}{N} \in \varphi_\tau(\mathbb{Z}^2),$$

and so $(\frac{c}{N}, \frac{d-1}{N})$ lies in $\mathbb{Z}^2$. We thus conclude that $c \equiv 0$ and $d \equiv 1$ modulo N. Since $ad - bc = 1$, we may also conclude that $a \equiv 1$ modulo N. This shows that the matrix A must lie in the set

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \pmod N, c \equiv 0 \pmod N \right\}.$$

Conversely, suppose $A \in \Gamma_1(N)$, and let $\tau' = A\tau$. Then $\frac{c}{N}\tau + \frac{d-1}{N} \in \varphi_\tau(\Lambda)$, so

$$(f_V^{A,\tau})^{-1}\left(\tfrac{1}{N} + \varphi_{\tau'}(\mathbb{Z}^2)\right) = (c\tau + d)\tfrac{1}{N} + \varphi_\tau(\mathbb{Z}^2) = \tfrac{1}{N} + \varphi_\tau(\mathbb{Z}^2).$$

Since by Theorem 3.10 we already know that the lattices $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau)$ and $(\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'})$ are isomorphic, this shows that $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$ and $(\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'}, [\frac{1}{N}])$ are isomorphic N-pointed lattices. We have thus proved the following analogue of Theorem 3.10.

Theorem 4.11. *Let $\tau, \tau' \in \mathfrak{h}$. Then the map $A \mapsto (f_V^{A,\tau}, f_\Lambda^A)$ gives a bijection*

$$\left\{ A \in \Gamma_1(N) \,\middle|\, A\tau = \tau' \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{isomorphisms of the } N\textit{-pointed lattices} \\ (\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}]) \textit{ and } (\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'}, [\frac{1}{N}]) \end{array} \right\}.$$

By Proposition 4.10, every N-pointed lattice of isomorphic to one of the form $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$, and by Theorem 4.11 the N-pointed lattices $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$ and $(\mathbb{C}, \mathbb{Z}^2, \varphi_{\tau'}, [\frac{1}{N}])$ are isomorphic if and only if τ and τ' lie in the same $\Gamma_1(N)$-orbit. Thus the space

$$Y_1(N) := \Gamma_1(N) \backslash \mathfrak{h}$$

classifies N-pointed lattices.

As mentioned, we want the action of $\Gamma_1(N)$ on $\mathfrak{h}$ to be free, so that the universal family of oriented based lattices on $\mathfrak{h}$ descends to a family of lattices on $Y_1(N)$. This is so when $N \geq 3$.

Proposition 4.12. *For all $N \geq 3$, $\Gamma_1(N)$ acts freely on $\mathfrak{h}$.*

Proof. Observe that $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is an element of $\Gamma_1(N)$, but $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is not. Also observe that $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is also not when $N \geq 3$. Thus none of the nonidentity matrices of Proposition 3.19 lie in $\Gamma_1(N)$ when $N \geq 3$, and so in these cases $\Gamma_1(N)$ acts freely on $\mathfrak{h}$. □

After our words of caution regarding automorphisms, the following corollary, in fact equivalent to the above proposition, should be of some comfort.

Corollary 4.13. *For $N \geq 3$, an N-pointed lattice has no nontrivial automorphisms.*

Proof. By Proposition 4.10, it suffices to show that N-pointed lattices of the form $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$ have no nontrivial automorphisms. By Theorem 4.11, all automorphisms of $(\mathbb{C}, \mathbb{Z}^2, \varphi_\tau, [\frac{1}{N}])$ are given by those $A \in \Gamma_1(N)$ such that $A\tau = \tau$. Proposition 4.12 shows the only such matrix is the identity. □

Modular Forms and Other Level Structures

The notations $Y_1(N)$ and $\Gamma_1(N)$ come from the theory of modular forms. A modular form of weight 0 is a meromorphic function on $\mathfrak{h}$ that is invariant under the action of $\mathrm{SL}(2, \mathbb{Z})$. More generally, a modular form of weight 0 and level N is a meromorphic function on $\mathfrak{h}$ that is invariant under the group

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2, \mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \pmod{N}, b, c \equiv 0 \pmod{N} \right\},$$

which we call the *principal congruence subgroup of level N*. We call any subgroup $\mathrm{SL}(2, \mathbb{Z})$ containing $\Gamma(N)$ a *congruence subgroup of level N*. The groups $\Gamma_1(N)$ are an example of such groups. Another class of important congruence subgroups are the groups

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2, \mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

Given a fixed integer N, the M-division points for all M dividing N form a group under addition modulo the lattice. This is in fact a $\mathbb{Z}/N\mathbb{Z}$-module, and it can be shown that a $\mathbb{Z}/N\mathbb{Z}$-basis for this module gives a level structure corresponding to the full congruence subgroup $\Gamma(N)$ in the same way that an N-division point corresponds to the group $\Gamma_1(N)$. Similarly, a subgroup of division points of order N corresponds to the group $\Gamma_0(N)$. See the books Husemöller [18, Ch.11] and Koblitz [20, Ch.III] for a discussion of the equivalent level structures on elliptic curves, and for more detail regarding their relationships with modular forms.

The quotient spaces $Y_1(N)$, $Y_0(N) := \Gamma_0(N)\backslash\mathfrak{h}$ and $Y(N) := \Gamma(N)\backslash\mathfrak{h}$ are known as *open modular curves*. These may be compactified in a natural fashion to form *modular curves*. It is a deep result that the modular curves $X_0(N)$—the compactifications of the $Y_0(N)$—play a role in classifying a subset of the elliptic curves over $\mathbb{C}$, known as the elliptic curves over $\mathbb{Q}$ in another, quite different, fashion: every elliptic curve over $\mathbb{Q}$ arises as the image of $X_0(N)$ for some N. This is known as the modularity theorem for elliptic curves over $\mathbb{Q}$. A few more details and further references can be found in Silverman [28, §C.13].

4.4 The Universal Family of N-pointed Lattices

In this section we use the example of N-pointed lattices to illustrate how the universal family of based lattices descends to a universal family for lattices with sufficiently rigid level structure. As we wish the action of $\Gamma_1(N)$ on $\mathfrak{h}$ to be free, we will assume throughout that $N \geq 3$.

We begin by defining what we mean by a family of N-pointed lattices.

Definitions 4.14. A *family of N-pointed lattices* $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$ is a family of lattices $(\mathcal{V}, \mathcal{L}, B, \Phi)$ together with a subset $\mathcal{P}$ of $\mathcal{V}$ such that there exists an open cover $\{U_\alpha\}$ of B and holomorphic sections $s_\alpha : U_\alpha \to \mathcal{V}_{U_\alpha}$ such that for all $x \in U_\alpha$ we have the equality of sets $\mathcal{P} \cap \mathcal{V}_x = s_\alpha(x) + \mathcal{L}_x$, and also such that for all $x \in B$ the fibre $(\mathcal{V}_x, \mathcal{L}_x, \Phi_x, \mathcal{P} \cap \mathcal{V}_x)$ is an N-pointed lattice. We shall write $\mathcal{P}_x$ for this intersection $\mathcal{P} \cap \mathcal{V}_x$. As usual we call a member U_α of such a cover a *trivialising neighbourhood* of $\mathcal{P}$, and we also call such a section s_α a *trivialising section* of $\mathcal{P}$. We also call $\mathcal{P}$ the *section of N-division points of the family.*

Given families of N-pointed lattices $(\mathcal{V}_1, \mathcal{L}_1, B, \Phi_1, \mathcal{P}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2, B, \Phi_2, \mathcal{P}_2)$ over the same base space B, a *morphism of families of N-pointed lattices* between these two families is a morphism of families of lattices (F_V, F_Λ) such that $F_V(\mathcal{P}_1) = \mathcal{P}_2$.

Example 4.15. Consider the subset of $\mathbb{C} \times \mathfrak{h}$ consisting of the union of the N-points associated to each based lattice in the universal family of oriented based lattices. This forms the set

$$[\![\tfrac{1}{N}]\!] := \{(\tfrac{1}{N} + m\tau + n, \tau) \in \mathbb{C} \times \tau \mid m, n \in \mathbb{Z}\}.$$

As $N[\![\frac{1}{N}]\!] \subset \Phi(\mathbb{Z}^2 \times \mathfrak{h})$ and $[\![\frac{1}{N}]\!]$ has a global trivialising section $s : \mathfrak{h} \to \mathbb{C} \times \mathfrak{h}; \tau \mapsto (\frac{1}{N}, \tau)$, the family $(\mathbb{C} \times \mathfrak{h}, \mathbb{Z}^2 \times \mathfrak{h}, \Phi, \mathfrak{h}, [\![\frac{1}{N}]\!])$ is a family of N-pointed lattices.

More generally, given any family $(\mathcal{V}, B, \Phi)$ of oriented based lattices, we may construct a family of N-pointed lattices by taking the fibrewise coset

$$[\![\tfrac{1}{N}\Phi(\{(0,1)\} \times B)]\!] := \{\tfrac{1}{N}\Phi((0,1), x) + \Phi((m,n), x) \in \mathcal{V} \mid x \in B,\ m, n \in \mathbb{Z}^2\}$$

to be the set of N-division points. Analogous to the case of a single lattice, we call this the family of N-pointed lattices *associated to* the given family of based lattices, and say that the family of based lattices *extends* the N-pointed family.

Observe that the set of N-division points $\mathcal{P}$ of a family of N-pointed lattices is itself a bundle over B with fibres isomorphic to Λ as sets. A moment's consideration, in particular taking into account the fact that the group structure on each fibre is preserved under pullbacks, shows that the properties of $\mathcal{P}$ are preserved under pullback by a holomorphic map. Thus the pullback of a family of N-pointed lattices along a holomorphic map is again a family of N-pointed lattices.

Analogously to the moduli problem for lattices, we define the moduli problem for N-pointed lattices as the problem of representing the functor that takes a complex manifold to the set of families of N-pointed lattices over it, and takes a holomorphic map of complex manifolds to the pullback map between sets of families. As for based lattices, we shall try to solve this problem by exhibiting a universal family. In order to construct this family, we now turn our attention to quotients of families of lattices.

Definition 4.16. Let Γ be a group acting on complex manifolds B, $\mathcal{V}$ and $\mathcal{L}$. We call a family of N-pointed lattices $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$ a Γ-*family of N-pointed lattices* if it is a Γ-family of lattices and the set $\mathcal{P}$ is closed under the action of Γ.

Suppose we have a Γ-family of N-pointed lattices. Since the section of N-division points $\mathcal{P}$ is closed under the action of Γ, the set $\Gamma\backslash\mathcal{P}$ is well-defined. Furthermore, it follows from the proof of Proposition 4.4 the fact that the quotient family looks locally like the given family, and so this set $\Gamma\backslash\mathcal{P}$ is a section of N-division points for the quotient family. Thus the quotient of a Γ-family of N-pointed lattices over B by Γ gives a family of N-pointed lattices over $\Gamma\backslash B$—in fact, it follows from Theorem 4.2 that Γ-families over B correspond precisely to families over $\Gamma\backslash B$.

Consider the family of N-pointed lattices $(\mathbb{C}\times\mathfrak{h}, \mathbb{Z}^2\times\mathfrak{h}, \mathfrak{h}, \Phi, [\![\frac{1}{N}]\!])$ constructed from the universal family of oriented based lattices. As the computation preceding the statement of Theorem 4.11 shows that the set $[\![\frac{1}{N}]\!]$ is invariant under the action of $\Gamma_1(N)$, this is a $\Gamma_1(N)$-family of N-pointed lattices. Recalling that $N \geq 3$, $\Gamma_1(N)$ acts freely and properly discontinuously on $\mathfrak{h}$, and so we may take the quotient. We write the quotient family

$$(\mathcal{V}^{(N)}, \mathcal{L}^{(N)}, Y_1(N), \Phi^{(N)}, \mathcal{P}^{(N)}).$$

We can now state the main theorem of this section.

Theorem 4.17. *Let $N \geq 3$. The family of N-pointed lattices*

$$(\mathcal{V}^{(N)}, \mathcal{L}^{(N)}, Y_1(N), \Phi^{(N)}, \mathcal{P}^{(N)})$$

is a universal family for N-pointed lattices.

A consequence of this that each $Y_1(N)$ is a fine moduli space for N-pointed lattices. Observe that $Y_1(3)$ is a fine moduli space for elliptic curves with some level structure, and is also an 8-sheeted cover of the coarse moduli space $\mathcal{M}$ for elliptic curves. In this sense it is a close approximation to a solution for the moduli problem for elliptic curves.

We shall prove this theorem by making use of the universal property of the universal family of oriented based lattices, and the correspondence between equivariant bundles and bundles on a quotient. Before beginning the proof proper, however, we first show how, given any family $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$ of N-pointed lattices, to construct a 'basisification' for it—a related family that is also a family of oriented based lattices—and discuss a few properties of this construction. To this end, define the set

$$\widetilde{B} := \left\{ (\ell_1, \ell_2) \in \mathcal{L} \times_B \mathcal{L} \;\middle|\; \begin{array}{c} \ell_1, \ell_2 \text{ is an oriented basis for } \mathcal{L}_{\pi_{\mathcal{L}}(\ell_1)} \\ \text{extending the } N\text{-division point } \mathcal{P}_{\pi_{\mathcal{L}}(\ell_1)} \end{array} \right\}.$$

To clarify, by an oriented basis for $\mathcal{L}_x$ we mean a basis ℓ_1, ℓ_2 for $\mathcal{L}_x$ such that the complex number $\tau \in \mathbb{C}$ for which $\Phi(\ell_1) = \tau\Phi(\ell_2)$ lies in the upper half plane $\mathfrak{h}$, and to say this basis extends the N-division point $\mathcal{P}_x$ is to say that $\frac{1}{N}\Phi(\ell_2)$ lies in $\mathcal{P}_x$. This set projects onto B by the map

$$\begin{aligned} \alpha : \widetilde{B} &\longrightarrow B; \\ (\ell_1, \ell_2) &\longmapsto \pi_{\mathcal{L}}(\ell_1). \end{aligned}$$

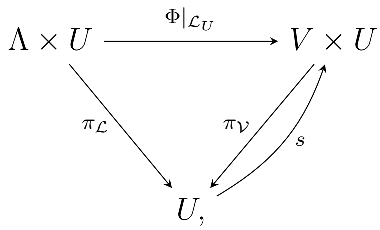
The key point of this construction is that the element of $\widetilde{B}$ are uniquely specified by a choice of point $x \in B$ and a choice of oriented basis for the fibre $\mathcal{L}_x$ of $\mathcal{L}$ over x that extends the N-point $\mathcal{P}_x$. Given $x \in B$, the preimage $\alpha^{-1}(x)$ indexes all possible oriented bases of the N-pointed lattice over x that agree with the N-point.

Observe that $\widetilde{B}$ is a subset of the manifold $\mathcal{L} \times \mathcal{L}$, and hence is endowed with the subspace topology. As α is the restriction of the composition of the projections $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and $\pi_{\mathcal{L}} : \mathcal{L} \to B$, it is continuous with respect to this topology. It is, in fact, a covering map.

Proposition 4.18. *Fix $N \geq 3$, and let $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$ be a family of N-pointed lattices over a complex manifold B. Then $\alpha : \widetilde{B} \to B$ is a covering of B.*

Proof. Fix $x_0 \in B$. To prove the proposition we need to find a open neighbourhood U of x_0 in B such that $\alpha^{-1}(U)$ consists of a collection of disjoint subsets of $\widetilde{B}$ each homeomorphic to U. Let U be a trivialising neighbourhood of x_0 of both the bundles $\mathcal{V}$ and $\mathcal{L}$ and also of the N-point $\mathcal{P}$—we shall say the U is a trivialising neighbourhood of the family $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$. I claim that U has the required property.

As U is a trivialising neighbourhood of $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$, we may write $\mathcal{L}_U = \Lambda \times U$, $\mathcal{V}_U = V \times U$, and depict the family over U as



where $s : U \to \mathcal{V}_U$ is a holomorphic section of $\pi_{\mathcal{V}}$ such that $\mathcal{P}_x = s(x) + \Phi(\mathcal{L}_x)$ for all $x \in U$. Note that $\mathcal{L}_U \times_B \mathcal{L}_U = \Lambda^2 \times U$. Thus

$$\begin{aligned}\alpha^{-1}(U) &= \left\{ (\ell_1, \ell_2) \in \mathcal{L}_U \times_B \mathcal{L}_U \;\middle|\; \begin{array}{c} \ell_1, \ell_2 \text{ is an oriented basis for } \mathcal{L}_{\pi_{\mathcal{L}}(\ell_1)} \\ \text{extending the } N\text{-division point } \mathcal{P}_{\pi_{\mathcal{L}}(\ell_1)} \end{array} \right\} \\ &= \left\{ (\lambda_1, \lambda_2, x) \in \Lambda^2 \times U \;\middle|\; \begin{array}{c} \lambda_1, \lambda_2 \text{ is an oriented basis for } \Lambda, \\ \Phi(\lambda_2, x) \in Ns(x) + N\Phi(\Lambda \times \{x\}) \end{array} \right\}.\end{aligned}$$

Observe now that, as $[s(x)]$ is an N-division point of the lattice over x, $Ns(x)$ lies in $\Phi(\Lambda \times \{x\})$. As Φ is injective, we may define its inverse function $\Phi^{-1} : \Phi(\Lambda \times U) \to \Lambda \times U$ on its image. Since $\pi_{\mathcal{V}}$ is holomorphic and $\pi_{\mathcal{L}}$ is a locally biholomorphic covering map, and the above diagram commutes, this function Φ^{-1} is holomorphic. In particular, Φ^{-1} is continuous on the image of Φ, and so $\Phi^{-1} \circ Ns$ gives a continuous section of $\pi_{\mathcal{L}} : \Lambda \times U \to U$. But Λ is discrete, so $\Phi^{-1} \circ Ns$ is the constant section $x \mapsto (\lambda, x)$ for some $\lambda \in \Lambda$. By composing the following expression with Φ^{-1}, we then see that

$$\Phi(\lambda_2, x) \in N\mathcal{P} = Ns(x) + N\Phi(\Lambda \times \{x\})$$

if and only if

$$(\lambda_2, x) \in (\lambda, x) + N\Lambda \times \{x\}.$$

Thus whether $\Phi(\lambda_2, x)$ lies in $N\mathcal{P}$ is independent of $x \in U$, and thus whether $(\lambda_1, \lambda_2, x)$ lies in $\alpha^{-1}(U)$ is independent of x. This shows that

$$\alpha^{-1}(U) = \{(\lambda_1, \lambda_2) \in \Lambda^2 \mid \lambda_1, \lambda_2 \text{ is an oriented basis for } \Lambda\} \times U.$$

Since Λ^2 is discrete, we conclude that α is a covering map. □

Since $\widetilde{B}$ is a cover of B, and B is a complex manifold, we may use the covering map α to give $\widetilde{B}$ the unique complex structure such that α is a locally biholomorphic covering map.

We now define a left $\Gamma_1(N)$-action on the space $\widetilde{B}$ as follows: for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ define the map

$$\begin{aligned} A : \widetilde{B} &\longrightarrow \widetilde{B}; \\ (\ell_1, \ell_2) &\longmapsto (\ell_1, \ell_2)A^T = (a\ell_1 + b\ell_2, c\ell_1 + d\ell_2). \end{aligned}$$

We check this map is well-defined for each $A \in \Gamma_1(N)$. Since $\Gamma_1(N)$ is contained in $\mathrm{SL}(2,\mathbb{Z})$, the resultant pair again gives a basis for $\mathcal{L}_{\pi_{\mathcal{L}}(\ell_1)}$. Also, since $\Phi(\ell_2)$ lies in $N\mathcal{P}$, the fibre $\mathcal{P}_{\pi_{\mathcal{L}}(\ell_1)}$ is a $\Phi(\mathcal{L}_{\pi_{\mathcal{L}}(\ell_1)})$-coset, and $\frac{c}{N}, \frac{d-1}{N}$ lie in $\mathbb{Z}$, we have

$$\Phi(c\ell_1 + d\ell_2) = \Phi(\ell_2) + N(\tfrac{c}{N}\Phi(\ell_1) + \tfrac{d-1}{N}\Phi(\ell_2)) \in N\mathcal{P}_{\pi_{\mathcal{L}}(\ell_1)} \subset N\mathcal{P}.$$

This shows this map is well-defined. Furthermore, as this map is simply multiplication on the left of the column vector $(\ell_1, \ell_2)^T$ by A, we have indeed defined a left action.

Let us consider this action for a moment. An element of $\widetilde{B}$ is specified by a choice of point $x \in B$ and a choice of oriented basis for the fibre $\mathcal{L}_x$ that extends the N-point $\mathcal{P}_x$. Given an element of $\Gamma_1(N)$, its action is to keep the point x fixed, but map the basis for the fibre $\mathcal{L}_x$ to another basis for $\mathcal{L}_x$ that also extends the N-point $\mathcal{P}_x$. By Theorem 4.11 this action is free and transitive on the set of such bases. This fact is the core of the following lemma.

Lemma 4.19. *The group $\Gamma_1(N)$ acts freely and properly discontinuously on $\widetilde{B}$. Moreover, the quotient $\Gamma_1(N)\backslash\widetilde{B}$ is biholomorphic to B via the map*

$$\begin{aligned} \beta : \Gamma_1(N)\backslash\widetilde{B} &\longrightarrow B; \\ [(\ell_1, \ell_2)] &\longmapsto \pi_{\mathcal{L}}(\ell_1). \end{aligned}$$

Proof. It is clear from the above discussion that this action is free.

To see that the action is properly discontinuous, we make use of the fact that α is a covering map. Choose distinct points (ℓ_1, ℓ_2), $(m_1, m_2) \in \widetilde{B}$. If $\pi_{\mathcal{L}}(\ell_1) \neq \pi_{\mathcal{L}}(m_1)$, then we may pick disjoint open sets U and V of B such that $\pi_{\mathcal{L}}(\ell_1) \in U$, $\pi_{\mathcal{L}}(m_1) \in V$. Since the action of $\Gamma_1(N)$ preserves the fibres of α, the sets $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are then open sets of $\widetilde{B}$ such that

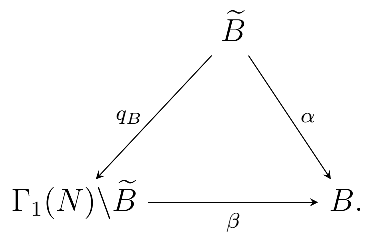
$$\{A \in \Gamma_1(N) \mid A\alpha^{-1}(U) \cap \alpha^{-1}(V) \neq \varnothing\} = \varnothing.$$

On the other hand, if $\pi_{\mathcal{L}}(\ell_1) = \pi_{\mathcal{L}}(m_1)$, then (ℓ_1, ℓ_2) and (m_1, m_2) lie in the same fibre of α over B. As α is a covering map, we may then pick disjoint open neighbourhoods U of (ℓ_1, ℓ_2) and V of (m_1, m_2) such that

$$\{A \in \Gamma_1(N) \mid A\alpha^{-1}(U) \cap \alpha^{-1}(V) \neq \varnothing\} = \{A \in \Gamma_1(N) \mid A(\ell_1, \ell_2) = (m_1, m_2)\}.$$

But the fibre of α over $\alpha(\ell_1, \ell_2) = \pi_{\mathcal{L}}(\ell_1)$ is the set of oriented bases of $\mathcal{L}_{\pi_{\mathcal{L}}(\ell_1)}$ that extend the N-point $\mathcal{P}_{\pi_{\mathcal{L}}(\ell_1)}$, and we have observed that $\Gamma_1(N)$ acts freely on this set. Thus $\{A \in \Gamma_1(N) \mid A(\ell_1, \ell_2) = (m_1, m_2)\}$ contains at most 1 element. This proves our action is properly discontinuous.

It remains to show that β is a biholomorphism. To begin observe that β is well-defined as the action of $\Gamma_1(N)$ on $\widetilde{B}$ preserves the fibres of $\mathcal{L}$ over B. Furthermore, the following triangle commutes:



Since both the vertical maps are locally biholomorphic covering maps, this shows that β is locally biholomorphic. But the orbits of $\Gamma_1(N)$ are precisely the sets of all oriented bases for a given fibre $\mathcal{L}_x$ of $\mathcal{L}$ that extend the N-point $\mathcal{P}_x$, so β is a bijection. This proves the theorem. □

Changing our notation slightly, we shall rename the map $\alpha : \widetilde{B} \to B$ as q_B, to reflect the fact we now know it is the quotient map induced by the $\Gamma_1(N)$-action. Making use of Theorem 4.2, the family $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$ of N-pointed lattices thus pulls back along q_B to a $\Gamma_1(N)$-family

$$(q_B^*\mathcal{V}, q_B^*\mathcal{L}, \widetilde{B}, q_B^*\Phi, q_B^*\mathcal{P})$$

of N-pointed lattices over $\widetilde{B}$. This pullback family can, as intended, naturally be given a global basis.

Proposition 4.20. *Fix $N \geq 3$, and let $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$ be a family of N-pointed lattices over a complex manifold B. Then the pullback family*

$$(q_B^*\mathcal{V}, q_B^*\mathcal{L}, \widetilde{B}, q_B^*\Phi, q_B^*\mathcal{P})$$

of N-pointed lattices of the map $q_B : \widetilde{B} \to B$ is extended to a family of oriented based lattices by the isomorphism

$$\begin{aligned} I : \mathbb{Z}^2 \times \widetilde{B} &\longrightarrow \mathcal{L} \times_B \widetilde{B} = q_B^* \mathcal{L}; \\ ((m,n),(\ell_1,\ell_2)) &\longmapsto (m\ell_1 + n\ell_2, (\ell_1,\ell_2)). \end{aligned}$$

Proof. To prove this proposition it suffices to show

$$\begin{aligned} s_1, s_2 : \widetilde{B} &\longrightarrow \mathcal{L} \times_B \widetilde{B}; \\ (\ell_1,\ell_2) &\overset{s_1}{\longmapsto} (\ell_1,(\ell_1,\ell_2)) \\ (\ell_1,\ell_2) &\overset{s_2}{\longmapsto} (\ell_2,(\ell_1,\ell_2)) \end{aligned}$$

are two global holomorphic sections of the Λ-fibred bundle $\mathcal{L} \times_B \widetilde{B}$ over $\widetilde{B}$ such that (i) on each fibre they form an oriented basis, and (ii) one Nth of the second section lies in $\mathcal{P} \times_B \widetilde{B}$. The property (i) ensures the map is an isomorphism, while the property (ii) ensures that the basis it gives is compatible with the N-division point on each fibre.

As the fibres of the Λ-fibred bundle $\mathcal{L}$ over B are discrete, the space $\mathcal{L} \times_B \widetilde{B}$ is a locally biholomorphic covering of $\widetilde{B}$, and so any section is holomorphic. The properties (i) and (ii) are an immediate consequence of the construction of $\widetilde{B}$: the points (ℓ_1,ℓ_2) of $\widetilde{B}$ are chosen precisely to have these properties. □

We shall write $\widetilde{\Phi} := q_B^*\Phi \circ I$, and hence write this family of oriented based lattices extending $(\mathcal{V},\mathcal{L},B,\Phi,\mathcal{P})$ as $(q_B^*\mathcal{V},\widetilde{B},\widetilde{\Phi})$. As $\mathfrak{h}$ is the universal family for families of oriented based lattices, this induces a map

$$\widetilde{T} : \widetilde{B} \longrightarrow \mathfrak{h}$$

such that this family of based lattices canonically isomorphic to the pullback along $\widetilde{T}$ of the universal family. This map takes $(\ell_1,\ell_2) \in \widetilde{B}$ to the complex number $\tau \in \mathfrak{h}$ such that $q_B^*\Phi(\ell_1,(\ell_1,\ell_2)) = \tau q_B^*\Phi(\ell_2,(\ell_1,\ell_2))$ in $q_B^*\mathcal{V}_{(\ell_1,\ell_2)}$, which is precisely the complex number such that $\Phi(\ell_1) = \tau\Phi(\ell_2)$ in $\mathcal{V}_{\pi_{\mathcal{L}}(\ell_1)}$.

As this is the isomorphism we wish to descend, it is important that the canonical isomorphism between the families of oriented based lattices $(q_B^*\mathcal{V},\widetilde{B},\widetilde{\Phi})$ and $(\widetilde{T}^*(\mathbb{C}\times\mathfrak{h}),\widetilde{B},\widetilde{T}^*\Omega)$ is in fact a $\Gamma_1(N)$-equivariant isomorphism. For this to even make sense, the pullback family $(\widetilde{T}^*(\mathbb{C}\times\mathfrak{h}),\widetilde{B},\widetilde{T}^*\Omega)$ must be a $\Gamma_1(N)$-family. This is implied by the following proposition.

Proposition 4.21. *The map $\widetilde{T} : \widetilde{B} \to \mathfrak{h}$ is $\Gamma_1(N)$-equivariant.*

Proof. Let $(\ell_1, \ell_2) \in \widetilde{B}$, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$. We wish to show that $A\widetilde{T}(\ell_1, \ell_2) = \widetilde{T}(A(\ell_1, \ell_2))$. Observe that $\widetilde{T}(\ell_1, \ell_2)$ is equal to complex number $\tau \in \mathfrak{h}$ such that $\Phi(\ell_1) = \tau\Phi(\ell_2)$, and that $\widetilde{T}(A(\ell_1, \ell_2)) = \widetilde{T}(a\ell_1 + b\ell_2, c\ell_1 + d\ell_2)$ is equal to the complex number $\tau' \in \mathfrak{h}$ such that $\Phi(a\ell_1 + b\ell_2) = \tau'\Phi(c\ell_1 + d\ell_2)$. We thus wish to show that $\tau' = \frac{a\tau+b}{c\tau+d}$.

Since Φ is a group map, we see that $a\Phi(\ell_1) + b\Phi(\ell_2) = \tau' c\Phi(\ell_1) + d\Phi(\ell_2)$. Substituting $\Phi(\ell_1) = \tau\Phi(\ell_2)$ into this, we have

$$a\tau\Phi(\ell_2) + b\Phi(\ell_2) = \tau'(c\tau\Phi(\ell_2) + d\Phi(\ell_2)).$$

Equating coefficients of $\Phi(\ell_2)$ and dividing through by $c\tau+d$ then gives $\tau' = \frac{a\tau+b}{c\tau+d}$, as required. □

By this proposition the pullback family $(\widetilde{T}^*(\mathbb{C} \times \mathfrak{h}), \widetilde{B}, \widetilde{T}^*\Omega)$ inherits a $\Gamma_1(N)$-action from the universal family $(\mathbb{C} \times \mathfrak{h}, \mathfrak{h}, \Omega)$ such that it is a $\Gamma_1(N)$-family.

Proposition 4.22. *The canonical isomorphism between the families of oriented based lattices* $(q_B^*\mathcal{V}, \widetilde{B}, \widetilde{\Phi})$ *and* $(\widetilde{T}^*(\mathbb{C} \times \mathfrak{h}), \widetilde{B}, \widetilde{T}^*\Omega)$ *given by the universal property of the family* $(\mathbb{C} \times \mathfrak{h}, \mathfrak{h}, \Omega)$ *is a* $\Gamma_1(N)$*-equivariant isomorphism of* $\Gamma_1(N)$*-families.*

Proof. We have the $\Gamma_1(N)$-families of oriented based lattices

$$\begin{array}{ccc} \mathbb{Z}^2 \times \widetilde{B} & \xrightarrow{\widetilde{\Phi}} & q_b^*\mathcal{V} \\ & \searrow \quad \swarrow & \\ & \widetilde{B} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{Z}^2 \times \widetilde{B} & \xrightarrow{\widetilde{T}^*\Omega} & \widetilde{T}^*(\mathbb{C} \times \mathfrak{h}) \\ & \searrow \quad \swarrow & \\ & \widetilde{B}. & \end{array}$$

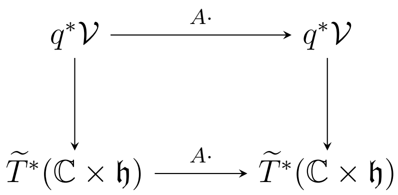
As $(\widetilde{T}^*(\mathbb{C} \times \mathfrak{h}), \widetilde{B}, \widetilde{T}^*\Omega)$ is the pullback of the universal family of oriented based lattices along the $\Gamma_1(N)$-equivariant map $\widetilde{T}$, the action of $\Gamma_1(N)$ on $\mathbb{Z}^2 \times \widetilde{B}$ in this $\Gamma_1(N)$-family is the familiar one: $A \in \Gamma_1(N)$ maps $((m, n), b)$ to $((m, n)A^{-1}, Ab)$.

On the other hand, observe that the action of $A \in \Gamma_1(N)$ on $\mathcal{L} \times_B \widetilde{B}$ takes $(\ell, (\ell_1, \ell_2))$ maps to $(\ell, (\ell_1, \ell_2)A^T)$. Writing ℓ as $(m, n)\left(\begin{smallmatrix} \ell_1 \\ \ell_2 \end{smallmatrix}\right)$, we may write this action as

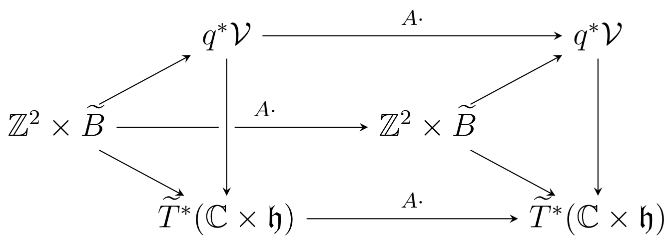
$$\begin{aligned} \big((m, n)\left(\begin{smallmatrix} \ell_1 \\ \ell_2 \end{smallmatrix}\right), (\ell_1, \ell_2)\big) \longmapsto & \big((m, n)\left(\begin{smallmatrix} \ell_1 \\ \ell_2 \end{smallmatrix}\right), (\ell_1, \ell_2)A^T\big) \\ &= \big((m, n)A^{-1}\big(A\left(\begin{smallmatrix} \ell_1 \\ \ell_2 \end{smallmatrix}\right)\big), (\ell_1, \ell_2)A^T\big) \end{aligned}$$

Applying the isomorphism I^{-1}, we thus see that the action of $A \in \Gamma_1(N)$ on $\mathbb{Z}^2 \times \widetilde{B}$ in the family $(q_B^*\mathcal{V}, \widetilde{B}, \widetilde{\Phi})$ is also the familiar one we have above.

Fix now $A \in \Gamma_1(N)$. We wish to show that



commutes, where the vertical maps are the canonical isomorphism. Note that the maps in this square are all $\mathbb{C}$-linear. Since these are $\Gamma_1(N)$-families and since $\Gamma_1(N)$ acts in the same way on $\mathbb{Z}^2 \times \widetilde{B}$ in both cases, we know that, with the possible exception of the above square, the following diagram commutes.



The aforementioned $\mathbb{C}$-linearity of the maps of the square in question then implies it does commute. □

Armed with the above propositions, we now proceed with the proof of the main theorem. We reiterate that the key idea is that the universal property of the universal family of oriented based lattices descends to the $\Gamma_1(N)$-quotient.

Proof of Theorem 4.17. We wish to show that for every family $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$ of N-pointed lattices over a complex manifold B, there exists a unique holomorphic map $T : B \to Y_1(N)$ such that this family is canonically isomorphic to the pullback of the family

$$(\mathcal{V}^{(N)}, \mathcal{L}^{(N)}, \Phi^{(N)}, Y_1(N), \mathcal{P}^{(N)})$$

along T.

Since the pullback family of N-pointed lattices along T is to be isomorphic to the given one, T must have the property that the fibre over $x \in B$ is isomorphic as an N-pointed lattice to the fibre over $T(x) \in Y_1(N)$. But as the family over $Y_1(N)$ contains exactly one representative from each isomorphism class of N-pointed lattices, this completely determines T. Thus, if T exists, T must be unique.

We now show T indeed exists. Fix the family $(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$. The construction of the map T is straightforward: Proposition 4.21 shows the holomorphic map $\widetilde{T} : \widetilde{B} \to \mathfrak{h}$ is $\Gamma_1(N)$-equivariant, and as $\Gamma_1(N)$ acts freely and properly discontinuously on both $\widetilde{B}$ and $\mathfrak{h}$, Theorem 4.4 shows it descends to a holomorphic map between their $\Gamma_1(N)$-quotients. By Lemma 4.19 and by definition, respectively, these quotients are precisely B and $Y_1(N)$. We thus have the commutative diagram

$$\begin{array}{ccc} \widetilde{B} & \xrightarrow{\widetilde{T}} & \mathfrak{h} \\ \downarrow{\scriptstyle q_B} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{T} & Y_1(N). \end{array}$$

Having constructed the map T, we now wish to show that it has the claimed property. That is, we want to show that

$$(\mathcal{V}, \mathcal{L}, B, \Phi, \mathcal{P})$$

and

$$(T^*\mathcal{V}^{(N)}, T^*\mathcal{L}^{(N)}, B, T^*\Phi^{(N)}, T^*\mathcal{P}^{(N)})$$

are canonically isomorphic.

By Theorem 4.2, it suffices to show that their pullbacks to $\widetilde{B}$

$$(q_B^*\mathcal{V}, q_B^*\mathcal{L}, \widetilde{B}, q_B^*\Phi, q_B^*\mathcal{P})$$

and

$$\big(q_B^*(T^*\mathcal{V}^{(N)}), q_B^*(T^*\mathcal{L}^{(N)}), \widetilde{B}, q_B^*(T^*\Phi^{(N)}), q_B^*(T^*\mathcal{P}^{(N)})\big)$$

are canonically isomorphic as $\Gamma_1(N)$-families. But by the commutativity of the above square, the second of these families is equal to

$$\big(\widetilde{T}^*(q^*\mathcal{V}^{(N)}), \widetilde{T}^*(q^*\mathcal{L}^{(N)}), \widetilde{B}, \widetilde{T}^*(q^*\Phi^{(N)}), \widetilde{T}^*(q^*\mathcal{P}^{(N)})\big),$$

and hence equal to

$$\big(\widetilde{T}^*(\mathbb{C} \times \mathfrak{h}), \widetilde{T}^*(\mathbb{Z}^2 \times \mathfrak{h}), \widetilde{B}, \widetilde{T}^*(\mathcal{U}), \widetilde{T}^*([\![\tfrac{1}{N}]\!])\big).$$

Note, however, that $\widetilde{T}$ is defined, using the universal property of the universal family of oriented based lattices, so that the families of oriented based lattices

$$(q_B^*\mathcal{V}, \widetilde{B}, \widetilde{\Phi})$$

and

$$\big(\widetilde{T}^*(\mathbb{C}\times\mathfrak{h}), \widetilde{B}, \widetilde{T}^*(\Omega)\big)$$

are canonically isomorphic. Proposition 4.22 shows that this is in fact a $\Gamma_1(N)$-equivariant isomorphism of $\Gamma_1(N)$-families. Furthermore, since these families extend the N-point structure given by $q_B^*\mathcal{P}$ and $\widetilde{T}^*([\![\frac{1}{N}]\!])$ respectively, this isomorphism is in fact a $\Gamma_1(N)$-equivariant isomorphism of the associated $\Gamma_1(N)$-families of N-pointed lattices

$$(q_B^*\mathcal{V}, q_B^*\mathcal{L}, \widetilde{B}, q_B^*\Phi, q_B^*\mathcal{P})$$

and

$$\big(q_B^*(T^*\mathcal{V}^{(N)}), q_B^*(T^*\mathcal{L}^{(N)}), \widetilde{B}, q_B^*(T^*\Phi^{(N)}), q_B^*(T^*\mathcal{P}^{(N)})\big).$$

This is what was required to prove the theorem. □

Concluding Remarks

In these pages we have introduced elliptic curves, as Riemann surfaces, and their families, with the aim of classifying all such families. To help with this we also introduced the language of moduli problems and moduli spaces. Through the use of some elementary Hodge theory, we then saw that the moduli problem for elliptic curves was equivalent to that for lattices, and proceeded to study the latter problem instead.

In the context of lattices, however, we saw that this problem had no solution as initially defined, and that the major barrier was the existence of nontrivial automorphisms of elliptic curves. Following this, we took up the study of based lattices—for which the moduli problem is solvable—and then examined their relationships with lattices. Relaxing the requirements for a moduli space, we used these ideas, and some results regarding the descent of complex structures, to provide a partial solution for the moduli problem for elliptic curves in the form of a coarse moduli space. In doing so we also came to an understanding of the automorphisms of lattices and elliptic curves.

We then turned our attention to a different method of providing a partial solution: that of level structures. By understanding the automorphisms of lattices we were able to construct structures on them that were not invariant under these automorphisms, and yet not too rigid, and thus construct a closely related, but solvable, moduli problem to that of lattices. Through studying the descent of families under quotients by a group action, we were then able to construct universal families for these objects, and solve the associated moduli problems.

In the end, though, one might still insist we want to construct some sort of universal family for elliptic curves—after all, the aim has been to classify families of elliptic curves, not just classify elliptic curves and their local deformations via a coarse moduli space, nor to classify families of some sort of slight rigidification of an elliptic curve. A final approach to consider then, is to relax what is meant by 'space' in the term moduli space, and look for solutions in some larger category.

While we have seen that there is no complex manifold over which lies a universal family of elliptic curves, it is possible to construct a universal family over what is known as an *orbifold*. Although there exists a universal family for based lattices over the upper half plane $\mathfrak{h}$, the existence of points with nontrivial stabilisers or, more importantly, stabilisers with order different to that of surrounding points, means that it does not descend to a universal family for lattices over its $\mathrm{SL}(2,\mathbb{Z})$-quotient $\mathcal{M}$. This lost data suggests we might want to view $\mathcal{M}$ as not just a complex manifold, but something whose local structure contains this information about how it is realised as a quotient of a disk by a finite group. This is, albeit roughly, the definition of an orbifold. A continuation of this conversation can be found in Hain [14, §3].

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